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# State-Set Observation of Uncertain Systems

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## 1 State observation

A dynamical system $\mathcal{S}$ is modelled in discrete-time:

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\,\boldsymbol{x}(k) + \boldsymbol{B}\,\boldsymbol{u}(k), \qquad (1)$$

$$\boldsymbol{y}(k) = \boldsymbol{C}\,\boldsymbol{x}(k) + \boldsymbol{D}\,\boldsymbol{u}(k). \qquad (2)$$

The aim of state observation is to reconstruct the value of the state vector $\boldsymbol{x}(k)$, $0 \leq k \leq \bar{k}$, given the model of the process and its input-output sequence over a given time horizon $[0, k]$:

$$\boldsymbol{U} = \boldsymbol{U}(0\ldots\bar{k}) = \big(\boldsymbol{u}(0), \boldsymbol{u}(1), \ldots, \boldsymbol{u}(\bar{k})\big),$$
$$\boldsymbol{Y} = \boldsymbol{Y}(0\ldots\bar{k}) = \big(\boldsymbol{y}(0), \boldsymbol{y}(1), \ldots, \boldsymbol{y}(\bar{k})\big).$$

This is described in [1] and shown in Fig. 1. The most common Luenberger observer is designed such that

$$\lim_{k\to\infty} \hat{\boldsymbol{x}}(k) = \boldsymbol{x}^{\circ}(k). \qquad (3)$$

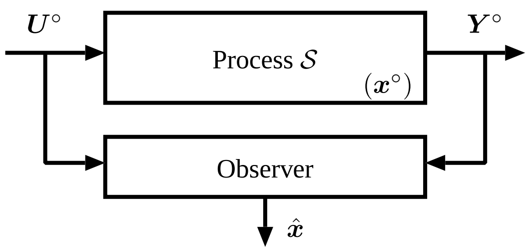



Figure 1: Observation of dynamical systems

## 2 Observation of uncertain systems

The state observation approach supposes an exact knowledge of both the model $\mathcal{S}$ and the input-output sequences. This is never the case in practice because the accuracy of process models is limited and measurements are affected by noise or measurement error. The process knowledge is *uncertain*.
At first, the project focuses on measurement errors. Thus, the true input and output sequences $(\boldsymbol{U}^{\circ}, \boldsymbol{Y}^{\circ})$ are unknown, and only biased sequences $(\tilde{\boldsymbol{U}}, \tilde{\boldsymbol{Y}})$ are measured. The error causes property (3) to never be attained. In this project an upper bounds for this error are assumed known:

$$|\tilde{\boldsymbol{u}}(k) - \boldsymbol{u}^{\circ}(k)| \leq \boldsymbol{e}_u(k) \quad \text{and} \quad |\tilde{\boldsymbol{y}}(k) - \boldsymbol{y}^{\circ}(k)| \leq \boldsymbol{e}_y(k).$$

Therefore, at each time step, two sets

$$\mathcal{U}(k) = \{\boldsymbol{u} \in \mathbb{R}^m \;|\; |\tilde{\boldsymbol{u}}(k) - \boldsymbol{u}| \leq \boldsymbol{e}_u(k)\} \qquad (4)$$

$$\text{and} \quad \mathcal{Y}(k) = \{\boldsymbol{y} \in \mathbb{R}^r \;|\; |\tilde{\boldsymbol{y}}(k) - \boldsymbol{y}| \leq \boldsymbol{e}_y(k)\} \qquad (5)$$

exist which are guaranteed to include the true input and output.

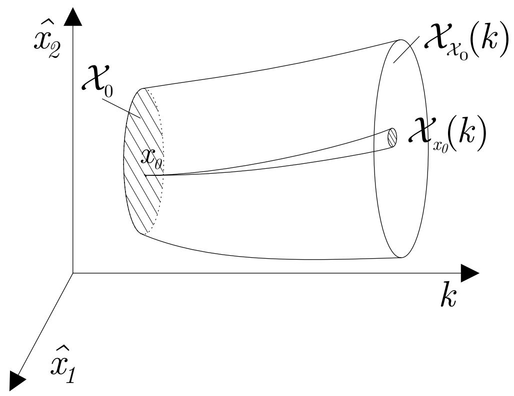



Figure 2: State trajectories for an uncertain system

As shown in Fig. 2, the state trajectory reconstructed for an uncertain system is not unique. Even if the initial state was exactly known (*i.e.* $\boldsymbol{x}(0) = \boldsymbol{x}_0$), input uncertainty allows for a bundle of state trajectories. After $k$ time steps, the state is only known to belong to a set $\mathcal{X}_{\boldsymbol{x}_0}(k)$. This uncertainty increases when the initial state is uncertain as well (*i.e.* $\boldsymbol{x}(0) \in \mathcal{X}_0$).
Furthermore, standard observers do not offer any information on the observation error $\boldsymbol{e}(k) = |\boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k)|$.

These drawbacks motivate the search of an alternative solution to the state observation of uncertain systems. As sets of input and output are considered, and because of the intrinsic characteristics of such systems, it is proposed to determine a set $\mathcal{X}$ of states which is guaranteed to include the true state:

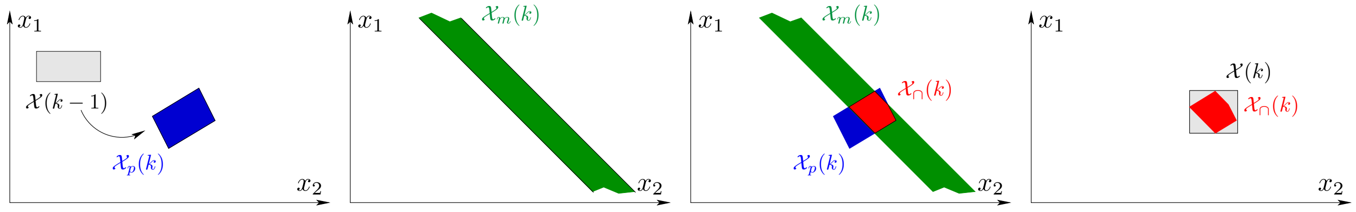
$$\boldsymbol{x}^{\circ}(k) \in \mathcal{X}(k). \qquad (6)$$

This task is referred to as a set-membership state observation, or *state-set observation*.

## 3 Algorithm for set observation

This section briefly describes the state-set observation algorithm used in this project. A graphical representation of the steps involved in this algorithm are shown in Fig. 3 for a second order system. A polytopic set representation was used for the implementation of the algorithm. Further details are found in [2].

**Given**:
- the model $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ of a process $\mathcal{S}$,
- measurements sequences $(\boldsymbol{U}, \boldsymbol{Y})(0\ldots\bar{k})$,
- uncertainty bounds on the input and output $(\boldsymbol{e}_u, \boldsymbol{e}_y)$.
- an a-priori state-set $\mathcal{X}_0$ (possibly $\mathcal{X}_0 = \mathbb{R}^n$),




(a) Predicted set (b) Measured set (c) Corrected set (d) Approximated set

Figure 3: Graphical representation of a polytopic state-set observation steps. Example of a second order system, $\boldsymbol{x} = (x_1, x_2)^T$.

Figure 4: Graphical representation of set observation steps

**Compute in a recursive loop** (initialise $k := 0$):

1. The input and output sets $\mathcal{U}(k)$, $\mathcal{Y}(k)$ from Eqs. (4)–(5)
2. The predicted set $\mathcal{X}_p(k)$
3. The measured set $\mathcal{X}_m(k)$
4. The corrected set $\mathcal{X}_\cap(k)$
5. The overapproximated set $\mathcal{X}(k)$
6. If $k < \bar{k}$, set $k := k+1$ and go to Step 1, otherwise exit the loop.

**Result**:
The state-sets $\mathcal{X}(k)$, $0 \le k \le \bar{k}$, verifying (6).

**Description of the computed sets**

The *predicted set* is computed using the Equation (1). It represents the set states $\boldsymbol{x}(k)$ reachable within one time step:

$$\mathcal{X}_p(k) = \big\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{A}\,\boldsymbol{x}' + \boldsymbol{B}\,\boldsymbol{u},\; \boldsymbol{x}' \in \mathcal{X}(k-1),\; \boldsymbol{u} \in \mathcal{U}(k-1)\big\}. \quad (7)$$

If $k = 0$, the set is initialised as: $\mathcal{X}_p(0) = \mathcal{X}_0$.
For the *measured set*, the output equation (2) is used to determine:

$$\mathcal{X}_m(k) = \big\{\boldsymbol{x} \in \mathbb{R}^n \mid \big(\boldsymbol{C}\,\boldsymbol{x} + \boldsymbol{D}\,\boldsymbol{u}\big) \in \mathcal{Y}(k), \boldsymbol{u} \in \mathcal{U}(k)\big\}. \quad (8)$$

The *corrected set* is the best possible approximation of the state-set at time $k$ and is obtained as the intersection:

$$\mathcal{X}_\cap(k) = \mathcal{X}_p(k) \cap \mathcal{X}_m(k). \quad (9)$$

While these first steps are theoretically sufficient to pursue a set observation, it is necessary in practise to add a step which keeps the computation burden constant in all recursions. Indeed, each intersection increases the complexity of representation of the state-set (additional faces appear as seen in Fig. 3(c)). This may be eluded by searching the smallest *overapproximated set* which wraps the corrected set into a simpler set (*e.g.* wrapping the set inside an interval box):

$$\mathcal{X}(k) \supseteq \mathcal{X}_\cap(k). \quad (10)$$

**Example of state-set observation**

A system of $5^{th}$ order is studied, for which the input and output sets are seen on the left side of Fig. 5. The sets $\mathcal{X}(k) \subset \mathbb{R}^5$ are projected along each state dimension in order to graphically represent them. This results, as seen in the right side of the figure, in confidence envelopes (solid lines) for each state coordinate which are guaranteed to contain the true state (dashed lines).

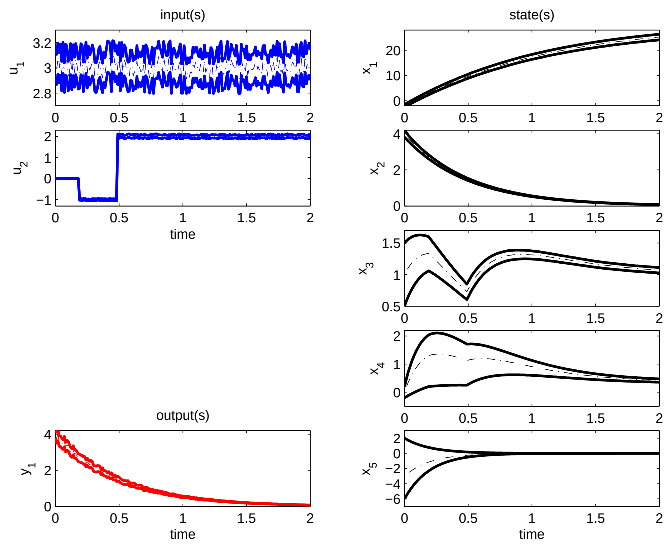



Figure 5: Example of a state-set observation

Current research interest lies in the analysis of set-observability criteria, and the achievable precision of the state estimation. A further research branch deals with the consistency-based diagnosis, using the set observer to test consistency of models and input-output measurements, see [3, 4, 5].

# References

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