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String stability for digitally networked platooning systems

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1 Introduction

Autonomous driving cars on highways is a goal for manufacturers for decades. Projects as SATRE by VOLVO or other projects financed by BMW, Mercedes-Benz, AUDI and others are an important research topic. It is known that already in 2020 first highways should be connected for testing. A lot of testing in restricted areas has been done, but it will take time to fulfill all political and safety requirements. A lot of research as been already published about cars driving in row with a constant distance d_0. A regular discussed topic is string stability. The fact that distance errors are not allowed to swing up in a platoon. A normal platoon of cars would use radar sensors to measure distances to the following car. These information can be used without a communication network to maintain a constant distance to the car in front of yours. Figure 1 presents a typical scenario.

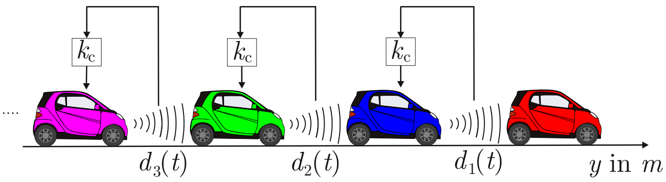

Figure 1: Platoon of cars with radar sensors

The cars, in this paper modeled as $\boldsymbol{IT}_1$ systems, receive the following distance to the following car inputs:

$$u_i(t) = k_{\mathrm{c}}\left(y_{i-1}(t) - y_i(t)\right) = k_{\mathrm{c}} d_{i-1}(t)$$

It is useful to look at distance error dynamics to reduce the platoon system matrix from n to $n-1$ subsystems and to focus on the differences in position:

$$\boldsymbol{e}_i(t) = \boldsymbol{x}_i(t) - \boldsymbol{x}_1(t)$$

A closed loop definition for the platoon without communication shows that everything reaches consensus, if $\boldsymbol{A} - \boldsymbol{b}k_{\mathrm{c}}\boldsymbol{c}^T$ is asymptotic stable.

$$\begin{pmatrix} \dot{\boldsymbol{e}}_2 \\ \dot{\boldsymbol{e}}_3 \\ \vdots \\ \dot{\boldsymbol{e}}_N \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} - \boldsymbol{b}k_{\mathrm{c}}\boldsymbol{c}^T & 0 & \cdots & 0 \\ 0 & \boldsymbol{A} - \boldsymbol{b}k_{\mathrm{c}}\boldsymbol{c}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{A} - \boldsymbol{b}k_{\mathrm{c}}\boldsymbol{c}^T \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \\ \vdots \\ \boldsymbol{e}_N \end{pmatrix}$$

State space model of used $\boldsymbol{IT}_1$ systems:

$$\dot{\boldsymbol{x}}_i(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{m} \end{pmatrix} \boldsymbol{x}_i(t) + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u_i(t), \qquad \boldsymbol{x}_i(0) = \begin{pmatrix} p_{0i} & v_{0i} \end{pmatrix}^T$$
$$y_i(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} \boldsymbol{x}_i(t) \qquad i \in \{1, ..., N\}$$

To guarantee string stability two conditions have to hold.

1) ***Stability:*** The closed loop agents have to be asymptotically stable. This is guaranteed with the system matrix to be Hurwitz stable.

2) ***Aperiodicity:*** Two equivalent statements are possible for second order car systems in the platoon.

   a)Impulse response $g_{i-1,i}(t) = \boldsymbol{c}^T e^{\tilde{\boldsymbol{A}}_{\boldsymbol{i}}\mathrm{t}} \boldsymbol{b} > 0 \quad \forall t > 0$ where $\tilde{\boldsymbol{A}}_{\boldsymbol{i}}$ is the closed loop system matrix.

   b) All eigenvalues are real and their static reinforcement k_s has to be greater than zero.

The subject of this project is summarised in the following problem statement:

Question: How is adding of communication structures changing string stability and the behaviour of the platoon.
Find: Analytic methods to analyse the platoon in respect to string stability.

2 Communication structure

A foresightful driver does not only look at the distance to the car in front of him, but additionally gather information about the cars further away. This information is essential, when situations on the road are changing drastically as in heavy traffic or traffic jams.

Figure 2 used the following input laws:

$$u_2 = k_1\left(y_1 - y_2\right) = k_1 d_1$$
$$u_l = k_{\mathrm{c}}\left(y_{l-1} - y_l\right) + k_{\mathrm{d}}\left(y_{l-2} - y_{l-1}\right) \quad \forall l > 2$$

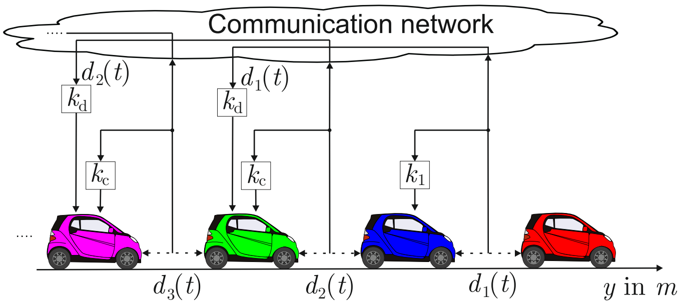

Figure 2: Platoon with communication network

The closed loop system has the following structure:

$$\begin{pmatrix}\dot{\boldsymbol{e}}_2\\ \dot{\boldsymbol{e}}_3\\ \dot{\boldsymbol{e}}_4\end{pmatrix} = \begin{pmatrix}\boldsymbol{A}-\boldsymbol{b}k_1\boldsymbol{c}^T & 0 & 0\\ \boldsymbol{b}\left(k_\mathrm{c}-k_\mathrm{d}\right)\boldsymbol{c}^T & \boldsymbol{A}-\boldsymbol{b}k_\mathrm{c}\boldsymbol{c}^T & 0\\ \boldsymbol{b}k_\mathrm{d}\boldsymbol{c}^T & \boldsymbol{b}\left(k_\mathrm{c}-k_\mathrm{d}\right)\boldsymbol{c}^T & \boldsymbol{A}-\boldsymbol{b}k_\mathrm{c}\boldsymbol{c}^T\end{pmatrix}\begin{pmatrix}\boldsymbol{e}_2\\ \boldsymbol{e}_3\\ \boldsymbol{e}_4\end{pmatrix}$$

This matrix is lower block triangular. It is easy to show that the eigenvalues of the system are the eigenvalues of the primary diagonal blocks. With this input law k_d has no influence on the eigenvalues of the system.
A new input law is introduced to show the similarities to common network structures of multi agent systems:

$$\begin{aligned} u_2 &= c_1\left(y_2-y_1\right) = -c_1 d_1\\ u_l &= -c_\mathrm{c}\left(y_l-y_{l-1}\right) - c_\mathrm{d}\left(y_l-y_{l-2}\right) \quad \forall l>2\end{aligned}$$

Both input laws are mathematically equivalent and the connections are:

$$c_1 = -k_1, \qquad c_\mathrm{d} = k_\mathrm{d}, \qquad c_\mathrm{c} = k_\mathrm{c} - k_\mathrm{d}.$$

This interpretation has the same practical value, because the on first sight unpractical term $(y_l - y_{l-2})$ is nothing else than an addition of $(y_l - y_{l-1})$ and $(y_{l-1} - y_{l-2})$ as used in the first approach. This could be easily done in the control system onboard.

The closed-loop system has a slightly different appearance:

$$\begin{pmatrix}\boldsymbol{A}+\boldsymbol{b}c_1\boldsymbol{c}^T & 0 & 0\\ \boldsymbol{b}c_\mathrm{c}\boldsymbol{c}^T & \boldsymbol{A}-\boldsymbol{b}\left(c_\mathrm{c}+c_\mathrm{d}\right)\boldsymbol{c}^T & 0\\ \boldsymbol{b}c_\mathrm{d}\boldsymbol{c}^T & \boldsymbol{b}c_\mathrm{c}\boldsymbol{c}^T & \boldsymbol{A}-\boldsymbol{b}\left(c_\mathrm{c}+c_\mathrm{d}\right)\boldsymbol{c}^T\end{pmatrix}$$

With this approach it is possible to change the eigenvalues of the system directly with the information received over the communication network.

3 Influence on string stability

In the last section a lot of emphasis was on changing the eigenvalues with the control parameters. In this section a derivation of string stability is shown additionally to changes occured by communication. Theory: For second order $\boldsymbol{IT}_1$ platooning systems to be string stable, it is sufficient that all eigenvalues are real and in the open left half plane.

If we consider the second control law the eigenvalues of each car after the second are:

$$\lambda_{1,2} = -\frac{1}{2m} \pm \sqrt{\frac{1}{4m^2} - \frac{c_\mathrm{c}+c_\mathrm{d}}{m}}$$

The proof is also valid for the first communication structure. Therefore an change of $c_\mathrm{c}+c_\mathrm{d} = k_\mathrm{c}$ can be used in the formulas. There are three cases to consider:

a) For $0 < c_\mathrm{c}+c_\mathrm{d} < \frac{1}{4m}$ both eigenvalues are real in the open left plane

b) For $c_\mathrm{c}+c_\mathrm{d} = \frac{1}{4m}$ there is a double eigenvalue $\lambda_{1,2} = \frac{-1}{2m}$

c) For $c_\mathrm{c}+c_\mathrm{d} > \frac{1}{4m}$ the eigenvalues $\lambda_{1,2}$ are complex, but still asymptotic stable

The eigenvectors form the tranformation matrix $\boldsymbol{V}$.

$$\boldsymbol{V} = \begin{pmatrix}1 & 1\\ \lambda_1 & \lambda_2\end{pmatrix}$$

The inverse exists for the discriminant not equal to zero $c_\mathrm{c}+c_\mathrm{d} \neq \frac{1}{4m}$. This special amplification means a double pole on the real axis before the eigenvalues follow the asymptotes parallel to the imaginary axis.

$$\boldsymbol{V}^{-1} = \begin{pmatrix}\frac{1}{2}\frac{1+\sqrt{-4(c_\mathrm{c}+c_\mathrm{d})m+1}}{\sqrt{-4(c_\mathrm{c}+c_\mathrm{d})m+1}} & \frac{m}{\sqrt{-4(c_\mathrm{c}+c_\mathrm{d})m+1}}\\ \frac{1}{2}\frac{-1+\sqrt{-4(c_\mathrm{c}+c_\mathrm{d})m+1}}{\sqrt{-4(c_\mathrm{c}+c_\mathrm{d})m+1}} & \frac{-m}{\sqrt{-4(c_\mathrm{c}+c_\mathrm{d})m+1}}\end{pmatrix}$$

It is known that the impulse response of a system has the following form:

$$\begin{aligned} g_{i-1,i}(t) &= \boldsymbol{c}^T e^{\tilde{\boldsymbol{A}}_i t}\boldsymbol{b}\\ &= \boldsymbol{c}^V \boldsymbol{V}\boldsymbol{V}^{-1} e^{\tilde{\boldsymbol{A}}_i t}\boldsymbol{V}\boldsymbol{V}^{-1}\boldsymbol{b}\\ &= \tilde{\boldsymbol{c}}^T\boldsymbol{V}^{-1} e^{\tilde{\boldsymbol{A}}_i t}\boldsymbol{V}\tilde{\boldsymbol{b}}\\ &= \tilde{\boldsymbol{c}}^T\boldsymbol{V}^{-1} \quad \text{diag}\; e^{\lambda_j t} \quad \boldsymbol{V}\tilde{\boldsymbol{b}}\end{aligned}$$

It follows that the impulse response is:

$$g_{i-1,i}(t) = \sum_{j=1}^{2} \tilde{c}_j^T \tilde{b}_j e^{\lambda_j t} = \sum_{j=1}^{2} g_j e^{\lambda_j t}$$

After calculus the impulse response of each car for $i > 2$ is:

$$g_{i-1,i}(t) = \frac{1}{\sqrt{-4\left(c_\mathrm{c}+c_\mathrm{d}\right)m+1}}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)$$

For every situation with two real unequal eigenvalues λ_1 is bigger than λ_2. It follows that $g_{i-1,i}(t) > 0 \quad \forall t > 0$. As the main result, string stability is given for case a) with the amplification bound $0 < c_\mathrm{c}+c_\mathrm{d} = k_\mathrm{c} < \frac{1}{4m}$.

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