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Design of optimal switching planes for discretely controlled continuous systems

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1 Introduction

Optimal control represents a major subject of classical control theory, primarily, because optimization provides an intuitive and systematic methodology for tuning controller parameter in situations, where the number of design freedoms largely exceeds the number required to meet the system specs. Recently, optimization was also heavily applied to switched and discretely controlled continuous systems (DCCS).
The latter constitutes a subclass of hybrid systems, where the state $\boldsymbol{x}(t)$ of a continuous plant is regulated by switching among discrete modes of operations $q \in \boldsymbol{Q}$ at appropriate instants of time $\bar{t}(k)$ (Fig. 1). A mode transition induces structural changes to the plant dynamics

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{d}(t), q(t)),\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t), \boldsymbol{d}(t), q(t)) \quad . \tag{2}$$

resulting in a different behavior at each mode, which is exploited to influence the future evolution of $\boldsymbol{x}(t)$. As an essential property, the functionality of a DCCS requires a prolonged switching action, such that at stationary operation, $\boldsymbol{x}(t)$ evolves either chaotically or periodically.

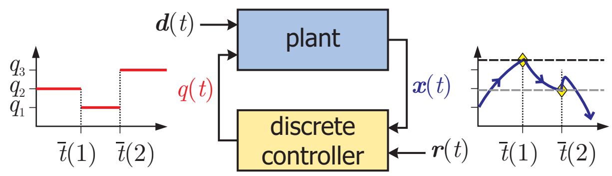

Figure 1: Structure of a discretely controlled continuous system

2 Project goals

Event-driven switching laws map into a static switching surface configurations in state space (Fig. 2). A mode transition is triggered, whenever the state $\boldsymbol{x}(t)$ intersects with one of these surfaces. The aim of this project is to *utilize standard optimization tools for a goal-oriented design of static switching planes of DCCS*. The considered task is, **given** the plant dynamics (1), (2) and a predetermined limit cycle Γ (see Fig. 2) to be executed at stationary operation, **find** a static switching plane configuration, which guarantees

1. local orbital stability of Γ,
2. a desired transient loop behavior and
3. robustness.

The main difficulty of optimal control is to reformulate the original problem as a meaningful constrained convex optimization problem. Whereas in classical control theory, optimality is typically related to minimizing the gain of a performance channel with physical meaning, defining optimality with respect to the orientation of switching planes is much less well understood and needs to be analyzed in initially.

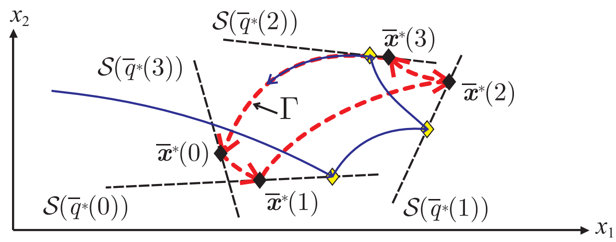

Figure 2: Stabilizing switching planes for a limit cycle Γ.

3 Design of optimal switching planes

In [2] it was shown that designing stabilizing switching planes

$$\mathcal{S}\left(\bar{q}_k^\star\right) = \left\{\boldsymbol{x} \mid \boldsymbol{n}^{\mathrm{T}}\left(\bar{q}_k^\star\right)\boldsymbol{x} - d\left(\bar{q}_k^\star\right) = 0\right\} \tag{3}$$

for a given Γ can be equivalently solved by designing a periodic state feedback controller for a constrained periodic system

$$\boldsymbol{\xi}(k+1) = \boldsymbol{A}(k)\boldsymbol{\xi}(k) + \boldsymbol{b}(k)u(k) \tag{4}$$

$$u(k) = -\boldsymbol{k}^{\mathrm{T}}(k)\boldsymbol{\xi}(k) \tag{5}$$

$$\boldsymbol{A}(k+p) = \boldsymbol{A}(k) = \partial\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+2}^\star\right) / \partial\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right) \tag{6}$$

$$\boldsymbol{b}(k+p) = \boldsymbol{b}(k) = \boldsymbol{A}(k)\boldsymbol{f}\left(\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right), \bar{q}_k^\star\right) \tag{7}$$

$$\forall \boldsymbol{k}^{\mathrm{T}}(k) \ :\ \boldsymbol{k}^{\mathrm{T}}(k)\boldsymbol{f}\left(\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right), \bar{q}_k^\star\right) = 1 \quad . \tag{8}$$

Thus, after some moderate extensions, all results of optimal control of discrete-time periodic systems (LQP-control) become available for solving the original problem. The classical LQP-problem setting (generalized H_2 problem) is as follows:

$$\text{Minimize } J = \frac{1}{2}\sum_{k=0}^{\infty}\left[\boldsymbol{\xi}^{\mathrm{T}}(k)\boldsymbol{Q}(k)\boldsymbol{\xi}(k) + u(k)ru(k)\right] \tag{9}$$

subject to:

1. $\boldsymbol{\xi}(k+1) = \boldsymbol{A}(k)\boldsymbol{\xi}(k) + \boldsymbol{b}(k)u(k)$ (10)
2. $u(k) = -\boldsymbol{k}^{\mathrm{T}}(k)\boldsymbol{\xi}(k)$ (11)
3. $\boldsymbol{P}(k) = \boldsymbol{P}(k+p) \succ 0$ (12)
4. $-\boldsymbol{P}(k) + (\bullet)^{\mathrm{T}}\boldsymbol{P}(k+1)\left(\boldsymbol{A}(k) - \boldsymbol{b}(k)\boldsymbol{k}^{\mathrm{T}}(k)\right) \preceq 0$ (13)

where J is the quadratic cost function with user-specified weights $\boldsymbol{Q}(k) = \boldsymbol{Q}(k{+}p) \geq 0$, $r(k) = r(k{+}p) \geq 0$. Although the analytic solution to (9)-(13), which involves a periodic Matrix-Ricatti-Equation

$$\boldsymbol{P}(k) = \boldsymbol{Q}(k) + \boldsymbol{A}^{\mathrm{T}}(k)\,\boldsymbol{P}(k{+}1)\,\boldsymbol{A}(k)$$
$$-\,[\bullet]^{\mathrm{T}}\left(r(k) + \boldsymbol{b}^{\mathrm{T}}(k)\,\boldsymbol{P}(k{+}1)\,\boldsymbol{b}(k)\right)^{-1}\left[\boldsymbol{b}^{\mathrm{T}}(k)\,\boldsymbol{P}(k{+}1)\,\boldsymbol{A}(k)\right] \ ,$$

provides valuable insight into the properties of the solution, it is beneficial to instead consider an equivalent LMI reformulation and solve this semidefinite program numerically by means of interior point methods.

Theorem 1. *Any solution $\boldsymbol{P}(k)$, $\boldsymbol{k}^{\mathrm{T}}(k)$ to the classical LQP-problem* (9)-(13) *for given weights $Qb(k)$, $r(k)$ is also a unique solution to the semidefinite program*

$$\textit{Minimize}\ \frac{1}{p}\sum_{k=0}^{p-1}\mathrm{tr}(\boldsymbol{W}(k)) \qquad (14)$$

subject to:

1. $\boldsymbol{\xi}(k{+}1) = \boldsymbol{A}(k)\boldsymbol{\xi} + \boldsymbol{b}(k)\,u(k) + \boldsymbol{b}_w(k)\,\boldsymbol{w}(k)$ (15)
2. $u(k) = -\boldsymbol{k}^{\mathrm{T}}(k)\,\boldsymbol{\xi}(k)$ (16)
3. $(\bullet)^{\mathrm{T}}\begin{pmatrix}\tilde{\boldsymbol{Q}}(k) & \tilde{r}(k)\end{pmatrix} = \begin{pmatrix}\boldsymbol{Q}(k) & \boldsymbol{0}\\ \boldsymbol{0}^{\mathrm{T}} & r(k)\end{pmatrix}$ (17)
4. $\begin{pmatrix}-\boldsymbol{X}(k) & \boldsymbol{A}(k)\,\boldsymbol{X}(k{+}1) + \boldsymbol{b}(k)\,\boldsymbol{l}^{\mathrm{T}}(k) & \boldsymbol{b}_w(k)\\ (\bullet)^{\mathrm{T}} & -\boldsymbol{X}(k{+}1) & \boldsymbol{0}\\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & -1\end{pmatrix} \leq 0$ (18)
5. $\begin{pmatrix}\boldsymbol{W}(k) & \tilde{\boldsymbol{Q}}(k) + \tilde{r}(k)\,\boldsymbol{l}^{\mathrm{T}}(k)\\ (\bullet)^{\mathrm{T}} & -\boldsymbol{X}(k{+}1)\end{pmatrix} \geq 0$ (19)

where $\boldsymbol{b}_w(k)$ can be chosen arbitrarily and the relations $\boldsymbol{P}(k) = \boldsymbol{X}^{-1}(k)$ and $\boldsymbol{k}^{\mathrm{T}}(k) = -\boldsymbol{k}^{\mathrm{T}}(k)\,\boldsymbol{X}^{-1}(k)$ hold for all k.

The advantage here is that the minimal constraint set (15)-(19) can be extended by additional convex constraints in order to account not only for orbital stability, but also for all other design objectives listed under Sec. 2.
To explicitly account for the algebraic constraints (8) the following result proves to be valuable.

Lemma 1. *The constraints* (8) *are satisfied, iff* $\det\left(\boldsymbol{A}(k) - \boldsymbol{b}(k)\,\boldsymbol{k}^{\mathrm{T}}(k)\right) = 0$ *holds for every k.*

It shows that any solution $\boldsymbol{k}^{\mathrm{T}}(k)$ to the LQP-problem satisfies (8) under the following necessary and sufficient condition.

Theorem 2. *A solution to the LQP-problem explicitly satisfies the constraints* (8)*, if and only if $r(k) = 0$, $\forall k$, whereas $\boldsymbol{Q}(k)$ may be chosen arbitrarily.*

4 Optimality criteria of DCCS

Since all desired loop properties of Sec. 2. uniquely map into convex constraints, there still exists a whole family of admissible switching plane configurations, provided that the constraint set is satisfiable at all. To find the best possible configuration among these, additional desired properties such as

- insensitivity to uncertainties in the event localization
- insensitivity to parameter variations in the plant dynamics
- a guaranteed mode transition in finite time
- a large region of attraction

need to be identified and related to a convex cost function.
A major source of disturbances are uncertainties in the event localization, which may have drastic effects on the orbital stability of Γ (see [1]). Since they cannot be avoided in a digital controller implementation, it is vital to minimize their influence.
Furthermore considerable parameter variations frequently occur in the plant, which should affect the stationary operation as little as possible. A plane configuration that robustly stabilizes Γ can be determined by solving a robust periodic state feedback problem. Keeping conservatism in the solution as low as possible requires a good and compact representation of the uncertain model. However, parameter variations also cause the limit cycle Γ to shift in the state space. The aim of avoiding large shifts must again be translated into a suitable cost criterion.
To explicitly consider the last two critical items in the design is indeed very complicated, as these cannot be represented by convex costs or constraints. Hence, suitable relaxations must be found.

5 Application example

The design of optimal switching planes can be applied to stabilize the periodic operation of a DC-DC boost converter in continuous conduction mode. The static switching surface design results in the switching planes depicted in Figure 3. These planes guarantee local orbital stability, a fast local transient response (multipliers magnitude $|m_i| < 0.3$) and minimize the sensitivity to event localization errors. The latter is achieve by maximizing the impact angle with the limit cycle Γ.

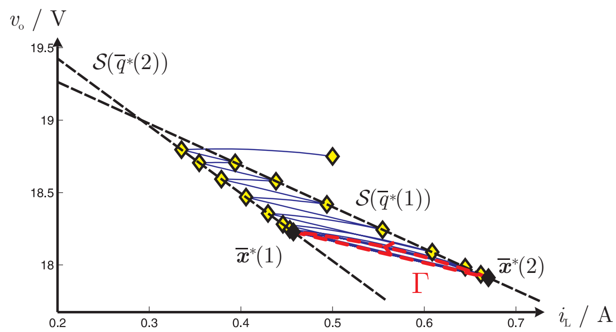

Figure 3: Stabilizing switching planes for a DC-DC boost converter minimizing the sensitivity to event localization errors.

References

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