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Stability analysis of event-based control loops

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1 Event-based state-feedback

The event-based control paradigm aims at reducing the usage of the feedback link within a control loop to time instants at which an event indicates the need for an information exchange between sensors, controller and actuators in order to retain a desired closed-loop performance. The current output measurements are used to update the control input only at the event times, denoted by t_k, where $k \in \mathbb{N}_0$ is the event counter.

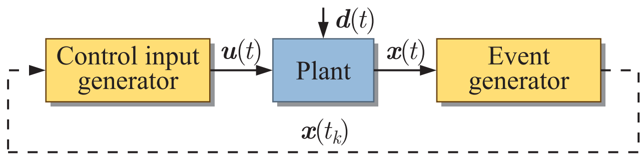


Figure 1: Event-based control loop

A state-feedback approach to event-based control has been introduced in [1]. The structure of this control scheme is illustrated in Fig. 1, where the dashed arrow indicates the feedback link that is only closed at $t = t_k (k = 0, 1, \ldots)$ and the solid lines represent continuous information transmission. The *plant* is described by the linear state-space model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{d}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \qquad (1)$$

where $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{d} \in \mathbb{R}^q$ denotes the state, input or disturbance, respectively. The *control input generator* applies a model of a continuous-time reference system in order to determine the control input $\boldsymbol{u}(t)$:

$$\dot{\boldsymbol{x}}_{\mathrm{s}}(t) = \bar{\boldsymbol{A}}\boldsymbol{x}_{\mathrm{s}}(t) + \boldsymbol{E}\hat{\boldsymbol{d}}_k, \quad \boldsymbol{x}_{\mathrm{s}}(t_k^+) = \boldsymbol{x}(t_k) \qquad (2)$$

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}_{\mathrm{s}}(t), \qquad (3)$$

where $\bar{\boldsymbol{A}} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})$. $\boldsymbol{x}_{\mathrm{s}} \in \mathbb{R}^n$ denotes the model state and $\hat{\boldsymbol{d}}_k \in \mathbb{R}^q$ a disturbance estimate. The feedback-gain $\boldsymbol{K}$ is designed so as to satisfy a desired disturbance rejection behavior in the continuous-time feedback system. The model state $\boldsymbol{x}_{\mathrm{s}}$ is reset to the current plant state $\boldsymbol{x}(t_k)$ each time when the *event generator* triggers an event, that is whenever the condition

$$\|\boldsymbol{x}(t_k) - \boldsymbol{x}_{\mathrm{s}}(t_k)\| = \bar{e} \qquad (4)$$

holds, where $\bar{e} \in \mathbb{R}_+$ is the event threshold.

2 Aim of the project

Two different methods for a stability analysis of the event-based state-feedback system have been derived:

- In [1] the stability of the event-based control loop has been proved by showing that the deviation between the behavior of the event-based and the (stable) continuous-time reference system is bounded.
- Reference [3] proposed a method for stability analysis in the sense of input-to-state stability (ISS) by modeling the error $\boldsymbol{x}_\Delta(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{\mathrm{s}}(t)$ as an input which is bounded due to the triggering condition (4).

However, both approaches did not explicitly take into account the hybrid nature of the closed-loop system.

The aim of this project is to develop a new method for the stability analysis considering the event-based control loop as a hybrid dynamical system, the behavior of which is characterized by a continuous flow between consecutive events and state resets at the event times t_k, [2]. This investigation shall also make clear relations between the novel analysis method and the existing results. In a second step the method shall be extended in order to make it applicable for the analysis of interconnected event-based control loops [4].

3 Stability analysis approach

The method for the stability analysis to be elaborated is based on the formulation of the event-based control system (1)–(4) as an impulsive system

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{x}}(t) = \hat{\boldsymbol{A}}\hat{\boldsymbol{x}}(t) + \hat{\boldsymbol{E}}\boldsymbol{d}(t), \qquad \text{for } \hat{\boldsymbol{x}}(t) \in \mathcal{C} \qquad (5a)$$

$$\hat{\boldsymbol{x}}(t^+) = \hat{\boldsymbol{G}}\hat{\boldsymbol{x}}(t), \qquad \text{for } \hat{\boldsymbol{x}}(t) \in \mathcal{D} \qquad (5b)$$

with $\hat{\boldsymbol{x}} = (\boldsymbol{x}^\top \ \boldsymbol{x}_{\mathrm{s}}^\top)^\top$ and

$$\hat{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{K} \\ \boldsymbol{0} & \bar{\boldsymbol{A}} \end{pmatrix}, \quad \hat{\boldsymbol{E}} = \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{0} \end{pmatrix}, \quad \hat{\boldsymbol{G}} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{I} & \boldsymbol{0} \end{pmatrix}.$$

Note that the disturbance estimate is set to $\hat{\boldsymbol{d}}_k \equiv \boldsymbol{0}$ here. The triggering condition (4) can be reformulated as

$$\hat{\boldsymbol{x}}^\top(t_k)\boldsymbol{Q}\hat{\boldsymbol{x}}(t_k) = \bar{e}^2 \quad \text{with} \quad \boldsymbol{Q} = \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix}$$

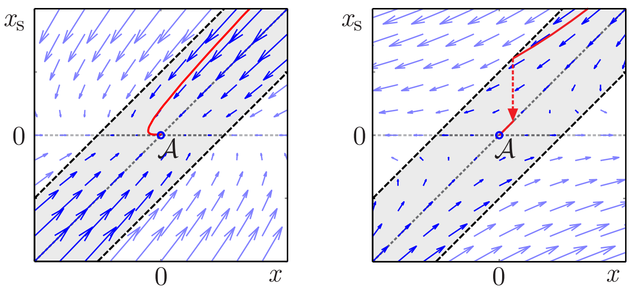
which implies the definition of the flow map $\mathcal{C}$ and reset map $\mathcal{D}$:

$$\mathcal{C} := \left\{ \hat{\boldsymbol{x}} \in \mathbb{R}^{2n} \mid \hat{\boldsymbol{x}}^\top \boldsymbol{Q} \hat{\boldsymbol{x}} < \bar{e}^2 \right\},$$
$$\mathcal{D} := \left\{ \hat{\boldsymbol{x}} \in \mathbb{R}^{2n} \mid \hat{\boldsymbol{x}}^\top \boldsymbol{Q} \hat{\boldsymbol{x}} = \bar{e}^2 \right\}.$$

The analysis aim is to prove the system (5a), (5b) to be ISS with respect to a *zero-invariant set* $\mathcal{A} \subset \mathbb{R}^{2n}$.

- The set $\mathcal{A}$ is said to be stable for the system (5a), (5b) with $\boldsymbol{d} = \boldsymbol{0}$ if for each $\varepsilon > 0$ there exists $\delta > 0$, so that $||\hat{\boldsymbol{x}}(0)||_{\mathcal{A}} < \delta$ implies $||\hat{\boldsymbol{x}}(t)||_{\mathcal{A}} < \varepsilon$. Here $||\hat{\boldsymbol{x}}||_{\mathcal{A}}$ denotes the point-to-set distance from $\hat{\boldsymbol{x}}$ to $\mathcal{A}$.
- The set $\mathcal{A}$ is said to be globally attractive if each solution $\hat{\boldsymbol{x}}(t)$ to the system (5a), (5b) with $\boldsymbol{d} = \boldsymbol{0}$ satisfies $||\hat{\boldsymbol{x}}(t)||_{\mathcal{A}} \to 0$ as $t \to \infty$.
- For the system (5a), (5b) with $\boldsymbol{d} = \boldsymbol{0}$ the set $\mathcal{A}$ is said to be globally asymptotically stable (0-GAS) if it is stable and globally attractive.

For the event-based state-feedback system the zero-invariant set $\mathcal{A}$ can be shown to be $\mathcal{A} = \{\boldsymbol{0}\}$. This is illustrated for a scalar system in Fig. 2 which shows the vector field of the flow dynamics (5a) for (a) a stable plant and (b) an unstable plant. The grey highlighted domain represents the flow set $\mathcal{C}$ and the black dashed lines the reset map $\mathcal{D}$.



(a) Stable plant (b) Unstable plant

Figure 2: Vector field of the undisturbed event-based control system (5a), (5b)

For continuous-time systems

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{d}(t)), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{6}$$

it is well known that the ISS-property is equivalent to 0-GAS and the asymptotic gain (AG) property.

- The system (6) is said to have an asymptotic gain if there exists $\gamma \in \mathcal{K}_\infty$ so that

$$\limsup_{t\to\infty} ||\boldsymbol{x}(t)||_{\mathcal{A}} \leq \gamma(||\boldsymbol{d}||_\infty) \tag{7}$$

  holds for all $\boldsymbol{x}_0$ and $\boldsymbol{d}$.

An essential problem to be solved in this project is to show that the equivalence between ISS and 0-GAS plus AG also holds for the impulsive system (5a), (5b), or respectively, to determine the conditions under which this equivalence is valid.

It is assumed that it is generally hard to find an AG for the system (5a), (5b) that satisfies the condition (7). Consider that an unstable plant implies instability of the flow dynamics (5a) of the event-based state-feedback loop (cf. Fig. 2(b)). Hence, the boundedness of the state $\hat{\boldsymbol{x}}(t)$ is due to the reinitialization of the model state $\boldsymbol{x}_s$ that occurs whenever $\hat{\boldsymbol{x}} \in \mathcal{D}$. This gives rise to the assumption that the AG for the impulsive system (5a), (5b) is not only a function of the disturbance $\boldsymbol{d}(t)$ but also depends upon the event threshold $\bar{e}$. The project has to answer the question how an AG for the impulsive system (5a), (5b) can be determined in dependence on the disturbance $\boldsymbol{d}(t)$ and the event threshold $\bar{e}$ and, moreover, how this gain is related to the results that are obtained by the existing analysis tools, e. g. the methods described in [3] or [4].

4 Extensions of the analysis

The analysis method that is developed for single-loop event-based control systems at first, is to be extended in a second step in two directions.

Consideration of the disturbance estimation. The impulsive system (5a), (5b) representing the behavior of the event-based state-feedback system does not include the disturbance estimation $\hat{\boldsymbol{d}}_k$ that can be applied to the model (2) and which might lead to large inter-event times $t_{\Delta k} = t_k - t_{k-1}$. A further investigation shall clarify how the analysis method must be extended in order to consider this estimate $\hat{\boldsymbol{d}}_k$ and how this extension affects the AG of the impulsive system.

Analysis of interconnected systems. While the investigations described so far only focus on single-loop event-based systems, the project shall also yield a method for the stability analysis of interconnected event-based control loops, as illustrated in Fig. 3.

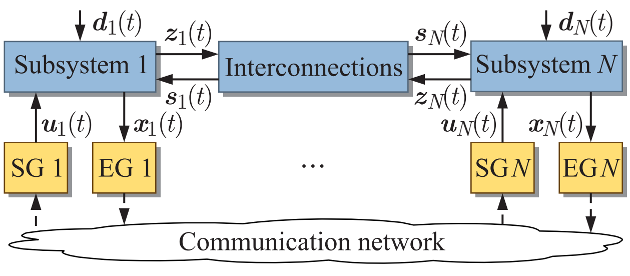


Figure 3: Event-based control of interconnected systems

References

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