Event-based control of nonlinear systems: An input-output linearization approach

Christian Stöcker and Jan Lunze

Abstract— **This paper proposes a new event-based control method for nonlinear SISO systems that are input-output linearizable and have internal dynamics. The main control objective is disturbance rejection while simultaneously reducing the feedback communication effort compared to a continuous control loop. The event-triggered control loop is shown to be ultimately bounded and, moreover, a bound for the deviation between this control loop and the continuous reference system is derived, which depends on the threshold of the event generator. Hence, by appropriately choosing the event threshold the event-based controller can be made to mimic the continuous control with desired accuracy. The novel control approach is evaluated by its application to a continuous stirred tank reactor.**

I. Introduction

A. Event-based control

Event-based control is a new control paradigm that aims at reducing the communication between the sensors, the controller and the actuators within a control loop by initiating a communication among these components only after an event has indicated that the control error exceeds a threshold. A potential application of this control strategy is in the field of networked control systems with intent to decrease the network utilization.

The structure of the event-based control loop that is investigated in this paper is depicted in Fig. 1. It consists of the following three components:

- the plant with single input $u(t)$, single output $y(t)$, state $\boldsymbol{x}(t)$ and disturbance $\boldsymbol{d}(t)$,
- the event generator and
- the control input generator, which incorporates the controller.

The solid arrows in Fig. 1 represent a continuous-time information transfer, whereas the dashed arrow indicates that this link is only used at the event times t_k $(k = 0, 1, ...)$. The event generator determines these event times t_k at which sensor data and previously processed signals like a disturbance estimation $\hat{\boldsymbol{d}}_k$ is fed back to the control input generator. The received information is used by the control input generator in order to update the trajectory of the control signal $u(t)$ for the time interval $t \in [t_k, t_{k+1})$.

This paper proposes a design method for the event-based control of nonlinear plants that is based on an input-output linearization approach. Following the idea of [6], the design

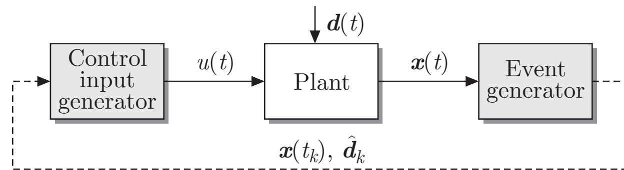

Fig. 1. Event-based control loop

aim is to make the event-based control loop mimic a continuous state-feedback loop, hereafter referred to as reference system, with prescribed accuracy. Copies of the reference system are used for the control input generation and the event generation. As the linearizing state feedback of the nonlinear plant is applied, the reference system is linear and so are the copies used in both generators. However, due to the disturbance $\boldsymbol{d}(t)$ and the event-based sampling, the generated control input differs from the linearizing input and the main analysis problem to be solved in this paper concerns the question how large the deviation of the event-based version of the feedback from its continuous counterpart is. An upper bound of this deviation is derived showing that the proposed event-based control method reaches the control aim.

B. Literature review

The literature on event-based control is predominantly focused on linear systems, whereas only a few publications investigate nonlinear systems. Event-based stabilization of nonlinear plants is studied in [11] using a Lyapunov-based technique. The idea of this approach is to approximate the derivative of the Lyapunov function for the continuously controlled system by a function of the plant state. The control input is updated each time this approximation reaches a nonnegative value. This basic idea has been extended in many ways, e. g. to distributed event-based control in [12] and event-based control with delays and data dropouts in [13].

An event trigger mechanism for the self-triggered stabilization of nonlinear plants is described in [9]. The predetermination of the next event time that has been investigated in this approach also relies on the knowledge of a Lyapunov function for the continuously controlled system. This technique has been refined for homogeneous and polynomial systems in [1].

Although the nonlinear plant is considered to be undisturbed in all mentioned publications the control input update is indispensable because a zero-order hold (ZOH) was used

This work was supported by the German Research Foundation (DFG) within the framework of the Priority Programme "Control Theory of Digitally Networked Dynamical Systems".

C. Stöcker and J. Lunze are with Faculty of Electrical Engineering and Information Sciences, Ruhr-University Bochum, 44780 Bochum, Germany {stoecker, lunze}@atp.rub.de

as control input generator that keeps the control signal constant between consecutive events.

[8] extended the work [6] to event-based disturbance rejection of input-output linearizable systems with relative degree $r = n$. This paper develops the control approach further to systems with internal dynamics where $r \leq n$. In contrast to the methods published in literature the event trigger mechanism does not depend on a Lyapunov function of the continuous closed-loop system. A smart control input generator is proposed instead of a ZOH, which generates an exponential control input signal.

C. Outline of the paper

Section II specifies the investigated class of nonlinear systems, details the control objective and introduces a reference system with ideal disturbance rejection behavior. A novel design method for the event generator and control input generator is proposed in Section III. Section IV proves the stability of the closed-loop system and analyzes the frequency of event generations. Section V provides an evaluation of the control approach by its simulative application to a chemical process.

D. Notation

The notation $|s|$ is used to denote the absolute value of a scalar s. $||\boldsymbol{x}(t)||$ denotes an arbitrary norm of an element $\boldsymbol{x} \in \mathbb{R}^n$ while $||\boldsymbol{x}(t)||_\infty$ refers to the supremum norm. A continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing and satisfies $\alpha(0) = 0$ and it is called of class $\mathcal{K}_\infty$ if it is unbounded. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is called of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each t and $\beta(r, t) \to 0$ as $t \to \infty$.

Given a function $\lambda(\boldsymbol{x})$ and a vector field $\boldsymbol{f}(\boldsymbol{x})$, then the derivative of λ along $\boldsymbol{f}$ (Lie derivative) is defined as

$$L_f \lambda(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i(\boldsymbol{x}).$$

The k-th derivative of λ along $\boldsymbol{f}$ is denoted by $L_f^k \lambda(\boldsymbol{x})$ and satisfies the recursion

$$L_f^k \lambda(\boldsymbol{x}) = \frac{\partial \left(L_f^{k-1} \lambda \right)}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x})$$

with $L_f^0 \lambda(\boldsymbol{x}) = \lambda(\boldsymbol{x})$.

II. Problem Statement

A. Plant and coordinates transformation

The plant is described by the nonlinear input-affine state-space model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}_{\mathrm{x}}(\boldsymbol{x}(t)) + \boldsymbol{g}_{\mathrm{x}}(\boldsymbol{x}(t)) u(t) + \boldsymbol{d}_{\mathrm{x}}(t), \;\; \boldsymbol{x}(0) = \boldsymbol{x}_0 \quad (1)$$

$$y(t) = h(\boldsymbol{x}(t)), \quad (2)$$

where $\boldsymbol{x} \in \mathbb{R}^n$ denotes the state vector, $u \in \mathbb{R}$ represents the input and $y \in \mathbb{R}$ the output. The disturbance is denoted by $\boldsymbol{d}_{\mathrm{x}} \in \mathcal{D}$ with $\mathcal{D}$ being a compact subset of $\mathbb{R}^n$ that contains the origin. $\boldsymbol{f}_{\mathrm{x}} : \mathbb{R}^n \to \mathbb{R}^n$ and $\boldsymbol{g}_{\mathrm{x}} : \mathbb{R}^n \to \mathbb{R}^n$ are continuous mappings and $\boldsymbol{f}_{\mathrm{x}}$ satisfies the relation $\boldsymbol{f}_{\mathrm{x}}(\boldsymbol{0}) = \boldsymbol{0}$. The plant state $\boldsymbol{x}(t)$ is assumed to be measurable.

Consider the plant (1) with the output (2) to have a well-defined relative degree $r \leq n$. The mapping

$$\boldsymbol{\phi}(\boldsymbol{x}(t)) = \begin{pmatrix} \phi_1(\boldsymbol{x}(t)) & \dots & \phi_n(\boldsymbol{x}(t)) \end{pmatrix}^{\mathrm{T}} \quad (3)$$

with

$$\phi_i(\boldsymbol{x}(t)) = L_{f_{\mathrm{x}}}^{i-1} h(\boldsymbol{x}(t)), \quad i = 1, ..., r$$

and the remaining functions $\phi_{r+1}(\boldsymbol{x}(t)), \dots, \phi_n(\boldsymbol{x}(t))$ chosen such that

$$L_{g_{\mathrm{x}}} \phi_i(\boldsymbol{x}(t)) = 0 \qquad \text{for all } r+1 \leq i \leq n$$

holds, qualifies as a transformation of the system (1), (2) into normal form with new coordinates $\boldsymbol{z}(t) = \boldsymbol{\phi}(\boldsymbol{x}(t))$. After defining

$$\boldsymbol{\xi}(t) = \begin{pmatrix} z_1(t) \dots z_r(t) \end{pmatrix}^{\mathrm{T}}, \quad \boldsymbol{\eta}(t) = \begin{pmatrix} z_{r+1}(t) \dots z_n(t) \end{pmatrix}^{\mathrm{T}},$$

the application of the transformation (3) to the system (1) yields the normal form

$$\dot{\boldsymbol{\xi}}(t) = \begin{pmatrix} z_2(t) \\ \vdots \\ z_r(t) \\ b(\boldsymbol{z}(t)) \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a(\boldsymbol{z}(t)) \end{pmatrix} u(t) + \boldsymbol{d}_\xi(t) \quad (4)$$

$$\dot{\boldsymbol{\eta}}(t) = \boldsymbol{q}(\boldsymbol{z}(t)) + \boldsymbol{d}_\eta(t) \quad (5)$$

with the transformed disturbance $\boldsymbol{d}(t) = \begin{pmatrix} \boldsymbol{d}_\xi^{\mathrm{T}}(t) & \boldsymbol{d}_\eta^{\mathrm{T}}(t) \end{pmatrix}^{\mathrm{T}}$ and the mapping $\boldsymbol{q} : \mathbb{R}^n \to \mathbb{R}^{n-r}$ given by

$$q_i(\boldsymbol{z}(t)) = L_f \phi_i(\boldsymbol{\phi}^{-1}(\boldsymbol{z}(t))) \qquad \text{for all } r+1 \leq i \leq n.$$

As proved in [5], such transformation exists for all input-affine systems (1).

The transformation of the nonlinear system (1) into the form (4), (5) obviously reveals the separation of the system into two coupled subsystems, which will subsequently be referred to as *input-output dynamics* (Eq. (4)) and *internal dynamics* (Eq. (5)). The internal dynamics (5) is supposed to be input-to-state stable (ISS). Thus, there exist functions $\theta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$, such that the solution to (5) is bounded by

$$||\boldsymbol{\eta}(t)|| \leq \theta(||\boldsymbol{\eta}(0)||, t) + \gamma_1(||\boldsymbol{\xi}||_\infty) + \gamma_2(||\boldsymbol{d}_\eta||_\infty). \quad (6)$$

B. Control objective

The investigated event-based control scheme aims at disturbance rejection in order to keep the plant state $\boldsymbol{z}(t)$ in a bounded surrounding of the setpoint $\overline{\boldsymbol{z}}$ (without loss of generality $\overline{\boldsymbol{z}} = \boldsymbol{0}$). This control objective is equivalent to the notion of ultimate boundedness [2], which means that the relation

$$\boldsymbol{z}(t) \in \Omega_{\mathrm{z}} \subset \mathbb{R}^n, \quad \forall t \geq 0 \quad (7)$$

holds for an appropriate set Ω_{z} satisfying $\overline{\boldsymbol{z}} \in \Omega_{\mathrm{z}}$. From the objective (7) it follows that the initial state $\boldsymbol{z}(0)$ is required to be contained in the set Ω_{z}.

C. Reference system

This section introduces a continuously controlled reference system that is deemed to have desired disturbance rejection behavior. The event-based control loop should mimic this continuous control loop. For linearizable plants, disturbance rejection can be accomplished by the linearizing state feedback

$$u(t) = \big(a(\boldsymbol{z}(t))\big)^{-1}\left(-b(\boldsymbol{z}(t)) - \boldsymbol{k}^{\mathrm{T}}\boldsymbol{\xi}(t)\right). \tag{8}$$

The application of the control input (8) to the plant (4), (5) results in the closed-loop system

$$\dot{\boldsymbol{\xi}}(t) = \boldsymbol{A}\boldsymbol{\xi}(t) + \boldsymbol{d}_{\xi}(t), \qquad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 \tag{9}$$

$$\dot{\boldsymbol{\eta}}(t) = \boldsymbol{q}(\boldsymbol{z}(t)) + \boldsymbol{d}_{\eta}(t), \qquad \boldsymbol{\eta}(0) = \boldsymbol{\eta}_0 \tag{10}$$

with quadratic r-dimensional matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -k_1 & -k_2 & \cdots & -k_r \end{pmatrix}. \tag{11}$$

In Eq. (11) $k_i > 0$ $(i = 1, ..., r)$ denotes the i-th element of the static state-feedback gain $\boldsymbol{k}^{\mathrm{T}}$, satisfying stability and disturbance rejection specifications for the closed-loop system (9), (10).

Since the reference system (9), (10) is ISS, the plant state $\boldsymbol{z}(t)$ is bounded by some functions $\theta_{\mathrm{r}} \in \mathcal{KL}$, $\gamma_{\mathrm{r}} \in \mathcal{K}_{\infty}$:

$$||\boldsymbol{z}(t)|| \leq \theta_{\mathrm{r}}(||\boldsymbol{z}(0)||\,, t) + \gamma_{\mathrm{r}}(||\boldsymbol{d}||_{\infty}).$$

Hence, the reference system (9), (10) is ultimately bounded with

$$\Omega_{\mathrm{z,r}} = \{\boldsymbol{z}|\, ||\boldsymbol{z}(t)|| \leq \theta_{\mathrm{r}}(||\boldsymbol{z}(0)||\,, t) + \gamma_{\mathrm{r}}(||\boldsymbol{d}||_{\infty})\}\,. \tag{12}$$

III. Nonlinear event-based control

This section proposes a design method for an event-based feedback that yields a closed-loop system with similar disturbance rejection behavior as the previously introduced reference system.

A. Control input generator

The control input generator applies a model of the reference system (9), (10), for which the state is denoted by $\boldsymbol{z}_{\mathrm{s}}(t) = \begin{pmatrix}\boldsymbol{\xi}_{\mathrm{s}}^{\mathrm{T}}(t) & \boldsymbol{\eta}_{\mathrm{s}}^{\mathrm{T}}(t)\end{pmatrix}^{\mathrm{T}}$, to determine the control input $u(t)$ according to

$$\dot{\boldsymbol{\xi}}_{\mathrm{s}}(t) = \boldsymbol{A}\boldsymbol{\xi}_{\mathrm{s}}(t) + \hat{\boldsymbol{d}}_{\xi,k}, \qquad \boldsymbol{\xi}_{\mathrm{s}}(t_k^+) = \boldsymbol{\xi}(t_k) \tag{13}$$

$$\dot{\boldsymbol{\eta}}_{\mathrm{s}}(t) = \boldsymbol{q}(\boldsymbol{z}_{\mathrm{s}}(t)) + \hat{\boldsymbol{d}}_{\eta,k}, \qquad \boldsymbol{\eta}_{\mathrm{s}}(t_k^+) = \boldsymbol{\eta}(t_k) \tag{14}$$

$$u(t) = \big(a(\boldsymbol{z}_{\mathrm{s}}(t))\big)^{-1}\left(-b(\boldsymbol{z}_{\mathrm{s}}(t)) - \boldsymbol{k}^{\mathrm{T}}\boldsymbol{\xi}_{\mathrm{s}}(t)\right). \tag{15}$$

Each event time t_k the control input generator receives the current plant state $\boldsymbol{z}(t_k)$ and reinitializes the model (13), (14). The time t_k^+ denotes the instant right after the event has occurred. Since the event generator has no information about the disturbance $\boldsymbol{d}(t)$ between consecutive events the input generation needs to rely on $\hat{\boldsymbol{d}}_{\xi,k}$ and $\hat{\boldsymbol{d}}_{\eta,k}$ which represent estimates of the disturbances $\boldsymbol{d}_{\xi}(t)$ and $\boldsymbol{d}_{\eta}(t)$, respectively for the time interval $t \in [t_k, t_{k+1})$. Note that this control approach works with an arbitrary disturbance estimation, including the trivial one ($\hat{\boldsymbol{d}}_k = \boldsymbol{0}$). An estimation method that is based on the assumption that the disturbance $\boldsymbol{d}(t)$ is a piecewise constant vector $\overline{\boldsymbol{d}}$ has been proposed in [8].

B. Event generator

The event generator indicates event times t_k at which a feedback is necessary. The following explains how these time instants are determined. Consider the plant (4), (5) with the control (15)

$$\dot{\boldsymbol{\xi}}(t) = \boldsymbol{A}\boldsymbol{\xi}(t) + \boldsymbol{e}_1\mu(\boldsymbol{z}(t), \boldsymbol{z}_{\mathrm{s}}(t)) + \boldsymbol{d}_{\xi}(t) \tag{16}$$

$$\dot{\boldsymbol{\eta}}(t) = \boldsymbol{q}(\boldsymbol{z}(t)) + \boldsymbol{d}_{\eta}(t) \tag{17}$$

with the r-dimensional vector $\boldsymbol{e}_1 = \begin{pmatrix}0 & \ldots & 0 & 1\end{pmatrix}^{\mathrm{T}}$ and

$$\mu(\boldsymbol{z}(t), \boldsymbol{z}_{\mathrm{s}}(t)) = \beta(\boldsymbol{z}(t), \boldsymbol{z}_{\mathrm{s}}(t)) + \boldsymbol{k}^{\mathrm{T}}\boldsymbol{\alpha}(\boldsymbol{z}(t), \boldsymbol{z}_{\mathrm{s}}(t)) \tag{18}$$

$$\beta(\boldsymbol{z}(t), \boldsymbol{z}_{\mathrm{s}}(t)) = b(\boldsymbol{z}(t)) - a(\boldsymbol{z}(t))\big(a(\boldsymbol{z}_{\mathrm{s}}(t))\big)^{-1}b(\boldsymbol{z}_{\mathrm{s}}(t))$$

$$\boldsymbol{\alpha}(\boldsymbol{z}(t), \boldsymbol{z}_{\mathrm{s}}(t)) = \boldsymbol{z}(t) - a(\boldsymbol{z}(t))\big(a(\boldsymbol{z}_{\mathrm{s}}(t))\big)^{-1}\boldsymbol{z}_{\mathrm{s}}(t).$$

A comparison of Eq. (16) with the input-output dynamics of the reference system (9) reveals that the ideal performance of the control loop is obtained for $\mu(\boldsymbol{z}(t), \boldsymbol{z}_{\mathrm{s}}(t)) = 0$. However, the fulfillment of this condition would require continuous state-feedback which is undesired in the event-based control scheme. Equation (18) can hence only be bounded according to $|\mu(\boldsymbol{z}(t), \boldsymbol{z}_{\mathrm{s}}(t))| \leq \bar{e}$ with $\bar{e} \in \mathbb{R}_+$ denoting the event threshold. An event is triggered whenever the relation

$$|\mu(\boldsymbol{z}(t), \boldsymbol{z}_{\mathrm{s}}(t))| = \bar{e} \tag{19}$$

holds which will subsequently be referred to as *trigger condition.*

Note that the control input $u(t)$, generated according to Eq. (15) is linearizing only if the model state $\boldsymbol{z}_{\mathrm{s}}(t)$ and the plant state $\boldsymbol{z}(t)$ coincide. Otherwise the signal $u(t)$ deviates from the linearizing input

$$u_{\mathrm{lin}}(t) = \big(a(\boldsymbol{z}(t))\big)^{-1}\left(b(\boldsymbol{z}(t)) - \boldsymbol{k}^{\mathrm{T}}\boldsymbol{\xi}(t)\right)$$

and the deviation error defined as $u_{\Delta}(t) = u(t) - u_{\mathrm{lin}}(t)$ is given by

$$u_{\Delta}(t) = \big(a(\boldsymbol{z}(t))\big)^{-1}\mu(\boldsymbol{z}(t), \boldsymbol{z}_{\mathrm{s}}(t)).$$

The last equation shows that the event function (18) correlates with deviation error $u_{\Delta}(t)$ of the input. This result can be used to specify the event threshold $\bar{e}$. It limits the deviation error since after each event $\mu(\boldsymbol{z}(t_k^+), \boldsymbol{z}_{\mathrm{s}}(t_k^+)) = 0$ holds due to the reinitialization.

C. Closed-loop system

In summary, the event based control loop consists of

- the plant (4), (5),
- the control input generator (13)–(15) and
- the event generator which triggers an event if the condition (19) is satisfied.

The generated event marks the time t_k at which the feedback is closed and the information $\boldsymbol{z}(t_k), \hat{\boldsymbol{d}}_k$ is communicated from the event generator to the control input generator.

IV. Analysis of the closed-loop system

A. Comparison of the event-based control loop and the reference system

This section shows that the difference between the behavior of the event-based control loop (16), (17) and of the reference system (9), (10) which is subsequently represented by the model

$$\dot{\boldsymbol{\xi}}_{\mathrm{r}}(t) = \boldsymbol{A}\boldsymbol{\xi}_{\mathrm{r}}(t) + \boldsymbol{d}_{\xi}(t) \quad (20)$$

$$\dot{\boldsymbol{\eta}}_{\mathrm{r}}(t) = \boldsymbol{q}(\boldsymbol{z}_{\mathrm{r}}(t)) + \boldsymbol{d}_{\eta}(t) \quad (21)$$

with state $\boldsymbol{z}_{\mathrm{r}}(t) = \begin{pmatrix}\boldsymbol{\xi}_{\mathrm{r}}^{\mathrm{T}}(t) & \boldsymbol{\eta}_{\mathrm{r}}^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}}$ is bounded from above by some bound that depends on the event threshold $\bar{e}$. Let

$$\boldsymbol{\delta}_{\xi}(t) = \boldsymbol{\xi}(t) - \boldsymbol{\xi}_{\mathrm{r}}(t) \quad (22)$$

$$\boldsymbol{\delta}_{\eta}(t) = \boldsymbol{\eta}(t) - \boldsymbol{\eta}_{\mathrm{r}}(t) \quad (23)$$

be the difference between the behavior of the event-based control system (16), (17) and the reference system (20), (21).

Theorem 1: The difference between the reference system (20), (21) and the event-based control loop (16), (17) is bounded from above by

$$\left|\left|\begin{pmatrix}\boldsymbol{\delta}_{\xi}(t)\\ \boldsymbol{\delta}_{\eta}(t)\end{pmatrix}\right|\right|_{\infty} \leq \delta_{\max} = \max\left\{\delta_{\xi\,\max}, \delta_{\eta\,\max}\right\} \quad (24)$$

with

$$\delta_{\xi\,\max} = \bar{e}\int_0^{\infty}\left|\left|\mathrm{e}^{\boldsymbol{A}\tau}\boldsymbol{e}_1\right|\right|_{\infty}\mathrm{d}\tau,$$

$$\delta_{\eta\,\max} = 2\big(\theta(||\boldsymbol{\eta}(0)||\,, t) + \gamma_2(||\boldsymbol{d}_{\eta}||_{\infty})\big) + \gamma_1(||\boldsymbol{\xi}||_{\infty}) + \gamma_1(||\boldsymbol{\xi}_{\mathrm{r}}||_{\infty}).$$

Proof: Equations (22), (23) are successively investigated with respect to the boundedness of $\boldsymbol{\delta}_{\xi}$ and $\boldsymbol{\delta}_{\eta}$, beginning with the difference (23) of the internal dynamics.

Recall that by assumption the boundedness of the internal dynamics is a property of the uncontrolled system according to (6). Since both the internal dynamics of the reference system (21) and of the event-based control loop (17) are ISS, their difference (23) is ISS as well and bounded by

$$||\boldsymbol{\delta}_{\eta}(t)|| \leq ||\boldsymbol{\eta}(t)|| + ||\boldsymbol{\eta}_{\mathrm{r}}(t)||\,.$$

Substitute (6) into this inequality yields

$$||\boldsymbol{\delta}_{\eta}(t)|| \leq 2\big(\theta(||\boldsymbol{\eta}(0)||\,, t) + \gamma_2(||\boldsymbol{d}_{\eta}||_{\infty})\big) + \gamma_1(||\boldsymbol{\xi}||_{\infty}) + \gamma_1(||\boldsymbol{\xi}_{\mathrm{r}}||_{\infty})$$

of which the right-hand side is denoted by $\delta_{\eta\,\max}$.

For the study of the difference behavior of the input-output dynamics, consider the system

$$\dot{\boldsymbol{\delta}}_{\xi}(t) = \boldsymbol{A}\boldsymbol{\delta}_{\xi}(t) + \boldsymbol{e}_1\mu(\boldsymbol{z}(t), \boldsymbol{z}_{\mathrm{s}}(t)), \quad \boldsymbol{\delta}_{\xi}(0) = \boldsymbol{0} \quad (25)$$

which follows from (16), (20), (22). The solution to (25) is given by

$$\boldsymbol{\delta}_{\xi}(t) = \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{e}_1\mu(\boldsymbol{z}(\tau), \boldsymbol{z}_{\mathrm{s}}(\tau))\mathrm{d}\tau.$$

An upper bound for this expression is obtained by the following estimation, exploiting the trigger condition (19):

$$||\boldsymbol{\delta}_{\xi}(t)||_{\infty} \leq \int_0^t \left|\left|\mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{e}_1\right|\right|_{\infty}|\mu(\boldsymbol{z}(\tau), \boldsymbol{z}_{\mathrm{s}}(\tau))|\,\mathrm{d}\tau$$
$$\leq \bar{e}\int_0^{\infty}\left|\left|\mathrm{e}^{\boldsymbol{A}\tau}\boldsymbol{e}_1\right|\right|_{\infty}\mathrm{d}\tau = \delta_{\xi\,\max}.$$

Since for the supremum norm

$$\left|\left|\begin{pmatrix}\boldsymbol{\delta}_{\xi}(t)\\ \boldsymbol{\delta}_{\eta}(t)\end{pmatrix}\right|\right|_{\infty} = \max\left\{||\boldsymbol{\delta}_{\xi}(t)||_{\infty}\,, ||\boldsymbol{\delta}_{\eta}(t)||_{\infty}\right\}$$

holds, the maximal deviation between the reference system (20), (21) and the event-based control loop (16), (17) is bounded according to (24). ■

The theorem shows that the state of the event-based control system always remains in a surrounding

$$\boldsymbol{z}(t) \in \Omega_{\delta}(\boldsymbol{z}_{\mathrm{r}}(t)) = \{\boldsymbol{z}(t)|\,||\boldsymbol{z}(t) - \boldsymbol{z}_{\mathrm{r}}(t)||_{\infty} \leq \delta_{\max}\}$$

of the reference system that is ultimately bounded. The event-based control loop is, hence, proved to be ultimately bounded, as well. As $\boldsymbol{z}_{\mathrm{r}}(t)$ remains in the set $\Omega_{\mathrm{z,r}}$ given by (12), the state $\boldsymbol{z}(t)$ of the event-based control loop remains in the set

$$\Omega_{\mathrm{z}} = \{\boldsymbol{z}|\,||\boldsymbol{z}|| \leq \theta(||\boldsymbol{z}_0||\,, t) + \gamma(||\boldsymbol{d}||_{\infty}) + \delta_{\max}\}$$

which shows the event-based control scheme to meet the control objective (7). Moreover, the deviation can be adjusted by appropriately setting the event threshold $\bar{e}$.

B. Communication frequency

This section studies the minimal inter-event time

$$T_{\min} = \arg\min_{t}\min_{\boldsymbol{z}(t_k)}\min_{\boldsymbol{d}(t)} \quad \text{s.t.}\ |\mu(\boldsymbol{z}(t), \boldsymbol{z}_{\mathrm{s}}(t))| = \bar{e} \quad (26)$$

on the assumption that the disturbance $\boldsymbol{d}(t)$ is bounded. Recall that the event function (18) grows due to a deviation between the plant state $\boldsymbol{z}(t)$ and the model state $\boldsymbol{z}_{\mathrm{s}}(t)$. Since (18) is nonlinear it reaches the event threshold $\bar{e}$ for different deviations

$$\boldsymbol{z}_{\Delta}(t) = \begin{pmatrix}\boldsymbol{\xi}_{\Delta}(t)\\ \boldsymbol{\eta}_{\Delta}(t)\end{pmatrix} = \boldsymbol{z}(t) - \boldsymbol{z}_{\mathrm{s}}(t). \quad (27)$$

The following investigation is based on the idea that for the whole set Ω_{z} there exists a minimal deviation $\boldsymbol{z}_{\Delta\,\min}$ for which the trigger condition (19) is satisfied (Fig. 2). It

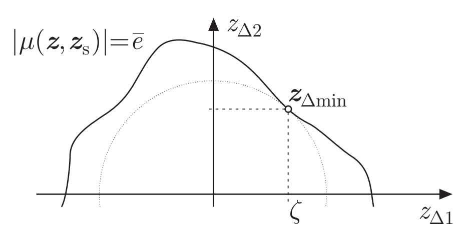

Fig. 2. Minimal deviation $\boldsymbol{z}_{\Delta\,\min}$ that leads to an event generation

exemplifies for a two-dimensional system the line on which $|\mu(\boldsymbol{z},\boldsymbol{z}_{\rm s})| = \overline{e}$ holds plotted against $\boldsymbol{z}_\Delta$. Define

$$\begin{aligned} \zeta := \min_{\boldsymbol{z},\boldsymbol{z}_{\rm s}} ||\boldsymbol{z}(t) - \boldsymbol{z}_{\rm s}(t)||_\infty \qquad & \text{for all } \boldsymbol{z} \in \Omega_{\rm z} \\ \text{s. t. } |\mu(\boldsymbol{z}(t),\boldsymbol{z}_{\rm s}(t))| = \overline{e} \end{aligned} \tag{28}$$

and note that ζ depends on the event threshold $\overline{e}$. In order to find the minimal inter-event time $T_{\min}$, the system

$$\dot{\boldsymbol{\xi}}_\Delta(t) = \boldsymbol{A}\boldsymbol{\xi}_\Delta(t) + \boldsymbol{e}_1\mu(\boldsymbol{z}(t),\boldsymbol{z}_{\rm s}(t)) + \boldsymbol{d}_{\xi\Delta}(t), \tag{29}$$

$$\dot{\boldsymbol{\eta}}_\Delta(t) = \boldsymbol{q}(\boldsymbol{z}(t)) - \boldsymbol{q}(\boldsymbol{z}_{\rm s}(t)) + \boldsymbol{d}_{\eta\Delta}(t), \tag{30}$$

$$\boldsymbol{\xi}_\Delta(t_k^+) = \boldsymbol{0}, \qquad \boldsymbol{\eta}_\Delta(t_k^+) = \boldsymbol{0}$$

is investigated, which follows from (13), (14) and (16), (17) and describes the dynamics of the difference state (27). The transformed disturbance

$$\boldsymbol{d}_\Delta(t) = \begin{pmatrix} \boldsymbol{d}_{\xi\Delta}(t) \\ \boldsymbol{d}_{\eta\Delta}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{d}_\xi(t) - \hat{\boldsymbol{d}}_{\xi,k} \\ \boldsymbol{d}_\eta(t) - \hat{\boldsymbol{d}}_{\eta,k} \end{pmatrix}$$

is assumed to be bounded by

$$||\boldsymbol{d}_{\xi\Delta}(t)|| \le \overline{d}_{\xi\Delta}, \quad ||\boldsymbol{d}_{\eta\Delta}(t)|| \le \overline{d}_{\eta\Delta}$$

for all $t \ge 0$. In the following the function $\boldsymbol{q}(\cdot)$ is considered to be Lipschitz with Lipschitz constant L

$$||\boldsymbol{q}(\boldsymbol{z}_1(t)) - \boldsymbol{q}(\boldsymbol{z}_2(t))|| \le L \cdot ||\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)||\,.$$

Theorem 2: The minimal time $T_{\min}$ between consecutive events is bounded from below $T_{\min} \ge \min\{\overline{T}_\xi, \overline{T}_\eta\}$ with $\overline{T}_\xi$ satisfying

$$\int_0^{\overline{T}_\xi} \left|\left|{\rm e}^{\boldsymbol{A}\tau}\right|\right|_\infty {\rm d}\tau = \frac{\zeta}{\overline{e} + \overline{d}_{\xi\Delta}} \tag{31}$$

and $\overline{T}_\eta$ given by

$$\overline{T}_\eta = \frac{\zeta}{L\cdot\zeta + \overline{d}_{\eta\Delta}}. \tag{32}$$

Proof: With (28) the definition (26) can be restated as

$$T_{\min} \ge \arg\min_t \min_{\boldsymbol{d}_\Delta(t)} \text{ s. t. } ||\boldsymbol{z}_\Delta(t)||_\infty = \zeta.$$

In what follows the problem of finding the minimal inter-event time $T_{\min}$ is separated into the tasks of determining the times

$$\begin{aligned} T_{\xi\min} &= \arg\min_t \min_{\boldsymbol{d}_{\xi\Delta}(t)} \text{ s. t. } ||\boldsymbol{\xi}_\Delta(t)||_\infty = \zeta, \\ T_{\eta\min} &= \arg\min_t \min_{\boldsymbol{d}_{\eta\Delta}(t)} \text{ s. t. } ||\boldsymbol{\eta}_\Delta(t)||_\infty = \zeta \end{aligned} \tag{33}$$

for which the relation

$$T_{\min} \ge \min\{T_{\xi\min}, T_{\eta\min}\}$$

holds. The following estimations develop lower bounds $\overline{T}_\xi$ and $\overline{T}_\eta$ on the times $T_{\xi\min}$ and $T_{\eta\min}$, respectively.

To begin with, consider the solution to (29)

$$\boldsymbol{\xi}_\Delta(t) = \int_{t_k}^t {\rm e}^{\boldsymbol{A}(t-\tau)}\Big(\boldsymbol{e}_1\mu(\boldsymbol{z}(\tau),\boldsymbol{z}_{\rm s}(\tau)) + \boldsymbol{d}_{\xi\Delta}(\tau)\Big){\rm d}\tau,$$

which is bounded by

$$||\boldsymbol{\xi}_\Delta(t)||_\infty \le \int_0^t \left|\left|{\rm e}^{\boldsymbol{A}\tau}\right|\right|_\infty {\rm d}\tau \left(\overline{e} + \overline{d}_{\xi\Delta}\right).$$

The minimal time for which the right-hand side of this inequality is equal to the value ζ is denoted by $\overline{T}_\xi$ and represents a lower bound on $T_{\xi\min}$. This time is obtained as the upper integral bound for which the relation (31) holds.

According to (33), $T_{\eta\min}$ is the minimal time for which the solution to (30) satisfies the equation

$$||\boldsymbol{\eta}_\Delta(t)|| = \left|\left|\int_0^t \Big(\boldsymbol{q}(\boldsymbol{z}(\tau)) - \boldsymbol{q}(\boldsymbol{z}_{\rm s}(\tau)) + \boldsymbol{d}_{\eta\Delta}(\tau)\Big){\rm d}\tau\right|\right| = \zeta.$$

An estimation of the left-hand side is obtained as follows:

$$\begin{aligned} ||\boldsymbol{\eta}_\Delta(t)|| &\le \int_0^t \Big(||\boldsymbol{q}(\boldsymbol{z}(\tau)) - \boldsymbol{q}(\boldsymbol{z}_{\rm s}(\tau))|| + ||\boldsymbol{d}_{\eta\Delta}(\tau)||\Big){\rm d}\tau \\ &\le \int_0^t \Big(L\cdot||\boldsymbol{z}(\tau) - \boldsymbol{z}_{\rm s}(\tau)|| + \overline{d}_{\eta\Delta}\Big){\rm d}\tau \\ &\le \int_0^t \Big(L\cdot\zeta + \overline{d}_{\eta\Delta}\Big){\rm d}\tau. \end{aligned} \tag{34}$$

The time for which (34) equals ζ is denoted by $\overline{T}_\eta$ and is given by (32). ■

V. Application Example

A. Continuously stirred tank reactor model

The event-based control approach is applied to control a chemical reaction in a continuous stirred tank reactor (CSTR), which is illustrated in Fig. 3. The tank is fed by a constant inflow q of the reactant A with temperature $\vartheta_{\rm in}$ and concentration $c_{\rm in}$. The temperature $\vartheta_{\rm c}$ of the cooling jacket is affected by the cooling power $\dot{Q}$ that serves as the input to the system. The liquid in the tank is supposed to be at a constant level. The reactions inside the liquid are described by the "van de Vusse" reaction scheme [10]

$$A \to B \to C, \quad 2A \to D$$

comprising the reaction of educt A to the desired product B and the parallel reactions to the undesired byproducts C and D. The liquid in the tank has the temperature ϑ. The concentration $c_{\rm in}$ and the temperature $\vartheta_{\rm in}$ of the inflow are subject to uncertainty and the deviations $\Delta c_{\rm in}$ and $\Delta\vartheta_{\rm in}$ from the nominal values are considered as disturbances of the

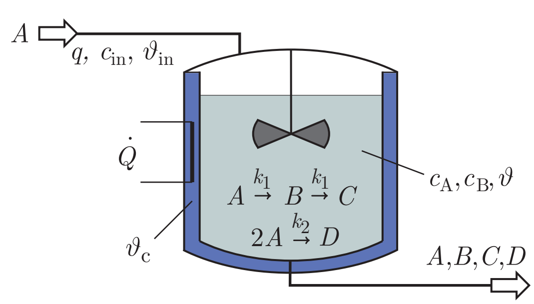

Fig. 3. Continuously stirred tank reactor

TABLE I

PARAMETERS OF THE CSTR MODEL (35)

Symbol	Value	Unit	Symbol	Value	Unit
κ_1	30.828	h^{-1}	κ_2	86.688	h^{-1}
κ_3	0.1	K/kJ	κ_4	$3.522 \cdot 10^{-4}$	m^3K/kJ
ϑ_{in}	104.9	°C	c_{in}	$5.1 \cdot 10^3$	mol/m^3
k_{10}	$1.287 \cdot 10^{12}$	h^{-1}	k_{20}	$9.043 \cdot 10^6$	m^3/(mol h)
E_1	9758.3	K	E_2	8560.0	K
ΔH_{AB}	4.2	kJ/mol	ΔH_{BC}	-11.0	kJ/mol
ΔH_{AD}	-41.85	kJ/mol	ϑ_{SP}	100	°C

process. The control aim is to keep the temperature ϑ of the liquid in the reactor in the setpoint ϑ_{SP}.

A CSTR of this type has been investigated in [4], according to which the dynamics of the chemical reaction inside the tank are described by the state-space model

$$\begin{pmatrix} \dot{c}_{\mathrm{A}} \\ \dot{c}_{\mathrm{B}} \\ \dot{\vartheta} \\ \dot{\vartheta}_{\mathrm{c}} \end{pmatrix} = \begin{pmatrix} -k_1(\vartheta)c_{\mathrm{A}} - k_2(\vartheta)c_{\mathrm{A}}^2 + (c_{\mathrm{in}} - c_{\mathrm{A}})\,q \\ k_1(\vartheta)c_{\mathrm{A}} - k_1(\vartheta)c_{\mathrm{B}} - c_{\mathrm{B}}q \\ h(c_{\mathrm{A}}, c_{\mathrm{B}}, \vartheta) + (\vartheta_{\mathrm{c}} - \vartheta)\kappa_1 + (\vartheta_{\mathrm{in}} - \vartheta)q \\ (\vartheta - \vartheta_{\mathrm{c}})\,\kappa_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \kappa_3 \end{pmatrix}^{\mathrm{T}} \dot{Q} + \begin{pmatrix} \Delta c_{\mathrm{in}} q & 0 & \Delta\vartheta_{\mathrm{in}} q & 0 \end{pmatrix}^{\mathrm{T}}, \quad (35)$$

where c_{A} and c_{B} denote the concentrations of the educt A and the product B, respectively. The temperature dependent reaction rates $k_1(\cdot)$ and $k_2(\cdot)$ are modeled with the Arrhenius function

$$k_i(\vartheta) = k_{i0}\, \exp\left(\frac{-E_i}{\vartheta + 273.15}\right), \quad i = 1, 2.$$

The reaction-induced change in temperature ϑ is described by

$$h(c_{\mathrm{A}}, c_{\mathrm{B}}, \vartheta) = -\kappa_4 \Big(k_1(\vartheta)\big(c_{\mathrm{A}}\Delta H_{\mathrm{AB}} + c_{\mathrm{B}}\Delta H_{\mathrm{BC}}\big) + k_2(\vartheta)c_{\mathrm{A}}^2 \Delta H_{\mathrm{AD}} \Big).$$

All other symbols and parameters are taken from [3] and [7] and are summarized in Table I. Taking the temperature ϑ as the output the system has the relative degree $r = 2$ and the model of the CSTR in normal form (4), (5) is obtained by use of the transformation

$$z_1 = \vartheta, \quad z_2 = \dot{\vartheta}, \quad z_3 = c_{\mathrm{A}}, \quad z_4 = c_{\mathrm{B}}$$

with $\boldsymbol{\xi} = \begin{pmatrix} z_1 & z_2 \end{pmatrix}^{\mathrm{T}}$ and $\boldsymbol{\eta} = \begin{pmatrix} z_3 & z_4 \end{pmatrix}^{\mathrm{T}}$.

B. Event-based control of the reactor temperature

The reference system (9), (10) with state-feedback gain $\boldsymbol{k}^{\mathrm{T}} = \begin{pmatrix} 18 & 9 \end{pmatrix}$ is defined to have satisfactory disturbance rejection behavior. This controller is, hence, applied in the control input generator and event generator, as well.

The following analysis investigates the event-based controlled system subject to a piecewise constant disturbance $\boldsymbol{d}(t)$. To begin with, the event threshold is set to $\bar{e} = 2 \cdot 10^4$. The simulation results for this setting are illustrated in Fig. 4. The first two subplots show the disturbances as dashed lines and the respective estimation derived with the method given in [8] as black solid lines. Subplots three and four depict the course of the reactor temperature ϑ and of the temperature ϑ_{c} of the cooling jacket. The dashed lines represent the behavior of the reference system and the solid lines the one of the event-based controlled system. The last subplot indicates the event times.

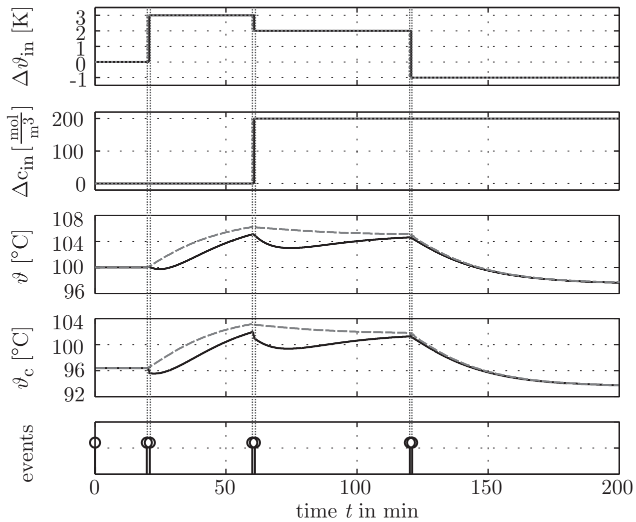

Fig. 4. Disturbance rejection behavior of the CSTR ($\bar{e} = 2 \cdot 10^4$)

At the beginning of the simulation the temperature and concentration of the inflow q does not deviate from the nominal values and the reactor temperature hence remains in the setpoint. At time $t = 20$ min the change of the inflow temperature triggers an event. After the disturbance has been estimated correctly at a second event at $t = 20.4$ min no further feedback is required until the temperature and concentration changes again at $t = 60$ min. During the simulation time of 200 minutes only seven events, including the initial one, are generated.

Figure 4 shows the behavior of the event-based controlled system to deviate noticeably from the one of the reference system. By decreasing the event threshold to $\bar{e} = 0.6 \cdot 10^4$ the event-based control loop can be made to mimic the continuously controlled behavior more accurate, as illustrated in Fig. 5 which shows the systems disturbance rejection behavior subject to the same disturbance as in the previous investigation. Note that in this case the same number of

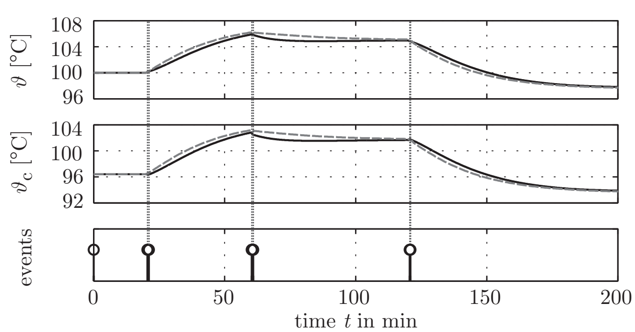

Fig. 5. Disturbance rejection behavior of the CSTR ($\bar{e} = 0.6 \cdot 10^4$)

events are generated despite a minor event threshold. The improved approximation of the behavior of the reference system comes at the cost of a shrinked time between consecutive events which is perceptible after each change of disturbance.

VI. Conclusion

The paper proposed a new event-based control scheme for nonlinear, input-output linearizable systems with internal dynamics. The deviation between the behavior of the event-based control loop and a continuous state-feedback reference system with ideal disturbance rejection was shown to be bounded. This bound can be made arbitrarily small by appropriately decreasing the event threshold. The event-based control scheme was proved to have a minimal time between consecutive events. An application example of the event-based control of the temperature in a continuously stirred tank reactor showed that the event-based control scheme works well in the sense that the frequency of feedback is considerable reduced, while a satisfactory disturbance rejection behavior is maintained.

References

[1] A. Anta and P. Tabuada, To sample or not to sample: Self-triggered control for nonlinear systems, *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2030–2042, 2010.

[2] F. Blanchini, Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions, *IEEE Transactions on Automatic Control*, vol. 39, no. 2, pp 428–433, 1994.

[3] H. Chen et al., Nonlinear predictive control of a benchmark CSTR, *Proceedings of 3rd European Control Conference*, pp. 3247–3252, 1995.

[4] S. Engell and K.-U. Klatt, Nonlinear control of a non-minimum-phase CSTR, *Proceedings of American Control Conference*, pp. 2941–2945, 1992.

[5] A. Isidori, *Nonlinear Control Systems*, Springer-Verlag, Heidelberg, 1995.

[6] J. Lunze and D. Lehmann, A state-feedback approach to event-based control, *Automatica*, vol. 46, no. 1, pp. 211–215, 2010.

[7] R. Rothfuss et al., Flatness based control of a nonlinear chemical reactor model, *Automatica*, vol. 32, no. 10, pp. 1433–1439, 1996.

[8] C. Stöcker and J. Lunze, Event-based control of input-output linearizable systems, *Proceedings of 18th IFAC World Congress*, accepted.

[9] P. Tabuada, Event-triggered real-time scheduling of stabilizing control tasks, *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1680–1685, 2007.

[10] J.G. van de Vusse. Plug-flow type reactor versus tank reactor. *Chemical Engineering Science*, vol. 19, no. 12, pp. 994–996, 1964.

[11] X. Wang and M. D. Lemmon, Event design in event-triggered feedback control systems. *Proceedings of 47th IEEE Conference on Decision and Control*, pp. 2105–2110, 2008.

[12] X. Wang and M. D. Lemmon, Event-triggered broadcasting feedback control systems, *Proceedings of American Control Conference*, pp. 3139–3144, 2008.

[13] X. Wang and M. D. Lemmon, Event-triggering n distributed networked systems with data dropouts and delays, *Proceedings of Hybrid Systems: Computation and Control*, pp. 366–380, 2009.