

# Input-to-state stability of event-based state-feedback control

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**Abstract**—This paper investigates the stability of event-based state-feedback loops. Two analysis methods are proposed. The first method proves the event-based state-feedback loop to be input-to-state practically stable which means that the state converges to a vicinity of the origin. The size of this region depends upon the magnitude of the disturbance and the event threshold, which is a design parameter. The second method is tailored for event-based state-feedback loops with stable plant dynamics and it shows that the investigated system is input-to-state stable, which implies that for small disturbance magnitudes the size of the region to which the state converges is independent of the event threshold. This new result shows that asymptotic stability can be achieved by means of an event-based controller with constant event threshold which has been proven in literature only for decreasing event thresholds. Both analysis methods are applied to a benchmark example and the results are compared with an analysis method known from literature which shows that the second proposed analysis method yields less conservative results with respect to the ultimate bound than existing methods in literature.

## I. INTRODUCTION

### A. Problem statement

The aim of event-based control is to restrict the feedback communication within a control loop to time instants at which an event indicates the need for an information exchange between sensors, controller and actuators in order to retain a desired quality of the closed-loop performance. This paper investigates the stability of event-based state-feedback loops that have the structure proposed in [9] (Fig. 1). The control input generator (CIG) incorporates a dynamic model of the plant with a continuous state feedback that generates the current control input  $u(t)$ . The event generator (EG) contains the same model and compares the state of this model with the continuously measured plant state  $x(t)$ . An event is triggered whenever the difference between both states reaches a defined threshold. At this event time  $t_k$  ( $k = 0, 1, \dots$ ), the current plant state  $x(t_k)$  is transmitted by the event generator to the control input generator and is used in both components in order to reset the model state.

In the original work [9], as well as in successive investigations like [7], [13], the event-based state-feedback loop has been analyzed by evaluating the system behavior in between consecutive events. The first aim of this paper is to develop a new analysis method that allows to uniformly investigate the system dynamics at and between event times. Inspired by [2], the event-based state-feedback loop is modeled as impulsive system, which is analyzed using techniques that have been elaborated in the theory on hybrid systems [8].

The investigated event-based state-feedback loop with stable plant dynamics is known not to generate any event if the disturbance  $d(t)$  is small or, moreover, to be asymptotically stable if the disturbance vanishes. This knowledge, however, is not reflected in the analysis methods in literature, e.g.

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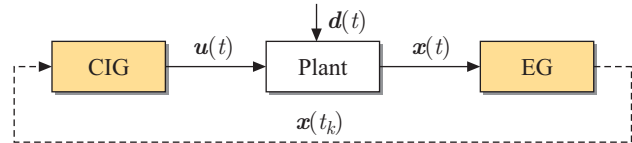


Fig. 1. Event-based state-feedback loop

[6]. The analysis methods published so far only prove the ultimate boundedness of the control loop but may not detect the asymptotic stability of the undisturbed loop. The second aim of this paper is to develop an analysis method that removes this conservatism of the existing methods.

### B. Literature review

In the literature on event-based control asymptotic stability of the closed-loop system has only been proven for approaches that use a decreasing event threshold. Design methods for triggering conditions that guarantee asymptotic convergence to the origin have been proposed in [10], [14]–[17] for continuous-time system and in [1] for discrete-time systems. In [3] asymptotic stability is obtained by means of a time-dependent decreasing event threshold. Stability analysis methods for event-based control approaches that use a constant event threshold have been published in [2], [9], [13] which, in contrast to the aforementioned works, result in ultimate boundedness rather than asymptotic stability.

Using an impulsive system formulation this paper shows that the event-based state-feedback loop, which uses a constant event threshold, is asymptotically stable if the plant has stable dynamics and the disturbance vanishes. This is a new result on event-based control, because asymptotic stability has been proven in the existing literature only for decreasing but not for constant event thresholds.

The impulsive system formulation of the event-based control system and its analysis with elaborated techniques has been introduced in [2]. In this reference a zero-order hold (ZOH) has been used for input generation. Therefore, the structure of the system investigated in [2] differs from the one that is studied in this paper where a dynamic input generator is applied. This difference will be explained in more detail in Sec. III-E.

### C. Outline of the paper

The model of the event-based state-feedback loop and its formulation as an impulsive system is given in Sec. II. Section III proposed a new Lyapunov-based stability analysis method that proves the event-based state-feedback loop to be input-to-state practically stable. An analysis method for event-based state-feedback loops with stable plant dynamics is presented in Sec. IV which, in extension of the previous result, shows that the control loop is input-to-state stable. Using the example of a thermofluid process, the results of both analysis methods are illustrated and compared to the analysis method proposed in [6] in Sec. V.

## D. Preliminaries

The following notation will be used. For a vector  $\mathbf{x} \in \mathbb{R}^n$ ,  $\|\mathbf{x}\|$  denotes its euclidean norm.  $\mathbf{x}(t^+) = \lim_{s \downarrow t} \mathbf{x}(s)$  represents the limit of  $\mathbf{x}(t)$  taken from above. For some signal  $\mathbf{d}(t)$ ,  $\|\mathbf{d}\|_\infty := \text{ess sup}_{t \geq 0} \|\mathbf{d}(t)\|$ .  $\mathbf{A}^\top \in \mathbb{R}^{m \times n}$  denotes the transpose of a matrix  $\mathbf{A} \in \mathbb{R}^{n \times m}$ .  $\lambda_{\min}(\mathbf{A})$  and  $\lambda_{\max}(\mathbf{A})$  refer to the minimum and maximum eigenvalue of a matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$ , respectively. For brevity a symmetric matrix  $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{C} \end{pmatrix}$  is written as  $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \star & \mathbf{C} \end{pmatrix}$ . A function  $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is of class  $\mathcal{K}$  if it is continuous, positive definite and strictly increasing and it is of class  $\mathcal{K}_\infty$  if it is also unbounded. A function  $\beta : \mathbb{R}_{> 0} \times \mathbb{R}_{> 0} \rightarrow \mathbb{R}_{> 0}$  is of class  $\mathcal{KL}$  if for each fixed  $t \geq 0$ ,  $\beta(\cdot, t)$  is of class  $\mathcal{K}$  and for each fixed  $s \geq 0$ ,  $\beta(s, t) \rightarrow 0$  as  $t \rightarrow \infty$ .

An impulsive system is represented by the model

$$\frac{d}{dt} \hat{\mathbf{x}}(t) = \hat{\mathbf{A}} \hat{\mathbf{x}}(t) + \hat{\mathbf{E}} \mathbf{d}(t), \quad \text{for } \hat{\mathbf{x}}(t) \in \mathcal{C}, \quad (1a)$$

$$\hat{\mathbf{x}}(t^+) = \hat{\mathbf{G}} \hat{\mathbf{x}}(t), \quad \text{for } \hat{\mathbf{x}}(t) \in \mathcal{D} \quad (1b)$$

with initial condition  $\hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0$  and where  $\hat{\mathbf{x}} \in \mathbb{R}^{\hat{n}}$  denotes the state and  $\mathbf{d} \in \mathbb{R}^p$  the disturbance. The sets  $\mathcal{C} \subset \mathbb{R}^{\hat{n}}$  and  $\mathcal{D} \subset \mathbb{R}^{\hat{n}}$  are referred to as flow set or reset set, respectively. The presented stability analysis methods for (1) refer to the following two notions of stability.

*Definition 1:* The system (1) is *input-to-state practically stable* (ISpS) if there exist functions  $\beta \in \mathcal{KL}$ ,  $\gamma \in \mathcal{K}_\infty$  and a scalar  $\sigma \in \mathbb{R}_{> 0}$  such that for all initial conditions  $\hat{\mathbf{x}}_0$  and every disturbance  $\mathbf{d}(t)$ , the solution to (1) exists and satisfies the relation

$$\|\hat{\mathbf{x}}(t)\| \leq \beta(\|\hat{\mathbf{x}}_0\|, t) + \gamma(\|\mathbf{d}\|_\infty) + \sigma, \quad \forall t \geq 0. \quad (2)$$

The system (1) is *input-to-state stable* (ISS) if the bound (2) holds with  $\sigma = 0$ .

*Definition 2:* The state  $\hat{\mathbf{x}}(t)$  of the system (1) is said to be *bounded* if (1) is either ISS or ISpS.

## II. EVENT-BASED STATE-FEEDBACK LOOP

### A. Model

In the investigated event-based state-feedback loop (Fig. 1) the plant is described by the linear model

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{E}\mathbf{d}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (3)$$

with plant state  $\mathbf{x} \in \mathbb{R}^n$ , control input  $\mathbf{u} \in \mathbb{R}^m$  and disturbance  $\mathbf{d} \in \mathbb{R}^p$  that is bounded by

$$\|\mathbf{d}(t)\| \leq \|\mathbf{d}\|_\infty = \bar{d}, \quad \forall t \geq 0. \quad (4)$$

The event-based state-feedback controller includes the control input generator (CIG) and the event generator (EG). The former is described by the model

$$\dot{\mathbf{x}}_s(t) = \bar{\mathbf{A}}\mathbf{x}_s(t), \quad \mathbf{x}_s(t_k^+) = \mathbf{x}(t_k), \quad (5a)$$

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}_s(t), \quad (5b)$$

where the feedback gain  $\mathbf{K}$  is chosen such that the matrix  $\bar{\mathbf{A}} = (\mathbf{A} - \mathbf{B}\mathbf{K})$  is Hurwitz.  $t_k$  ( $k = 0, 1, \dots$ ) with  $t_0 = 0$  denotes the event time at which the model state  $\mathbf{x}_s \in \mathbb{R}^n$  is reset to the current plant state  $\mathbf{x}(t_k)$ . These event times are determined by the event generator, which observes the difference state  $\mathbf{x}_\Delta(t) = \mathbf{x}(t) - \mathbf{x}_s(t)$  and triggers an event at time  $t$  if the condition

$$\|\mathbf{x}_\Delta(t)\| = \|\mathbf{x}(t) - \mathbf{x}_s(t)\| < \bar{e} \quad (6)$$

is violated, where the event threshold  $\bar{e} \in \mathbb{R}_{> 0}$  is a design parameter. At the event times  $t_k$  the current plant state  $\mathbf{x}(t_k)$  is transmitted to the control input generator.

### B. Impulsive system formulation

It has been shown in [9] that the event-based state-feedback loop (3)–(6) can be transformed into an equivalent representation that encompasses the controlled plant

$$\Sigma_c : \begin{cases} \dot{\mathbf{x}}(t) = \bar{\mathbf{A}}\mathbf{x}(t) + \mathbf{B}\mathbf{K}\mathbf{x}_\Delta(t) + \mathbf{E}\mathbf{d}(t), \\ \mathbf{x}(t_k^+) = \mathbf{x}(t_k) \end{cases} \quad (7)$$

together with the difference system

$$\Sigma_d : \begin{cases} \dot{\mathbf{x}}_\Delta(t) = \mathbf{A}\mathbf{x}_\Delta(t) + \mathbf{E}\mathbf{d}(t), \\ \mathbf{x}_\Delta(t_k^+) = \mathbf{0}. \end{cases} \quad (8)$$

With the state  $\hat{\mathbf{x}} = (\mathbf{x}^\top \ \mathbf{x}_\Delta^\top)^\top$ , Eqs. (7) and (8) form an impulsive system (1) with

$$\hat{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{A}} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} \end{pmatrix}, \quad \hat{\mathbf{E}} = \begin{pmatrix} \mathbf{E} \\ \mathbf{E} \end{pmatrix}, \quad \hat{\mathbf{G}} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \quad (9)$$

The event condition (6) can be restated as

$$\|\mathbf{x}_\Delta(t)\|^2 = \hat{\mathbf{x}}^\top(t) \mathbf{Q} \hat{\mathbf{x}}(t) < \bar{e}^2, \quad \mathbf{Q} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \quad (10)$$

which leads to the following definition of the flow set  $\mathcal{C}$  and reset set  $\mathcal{D}$ :

$$\mathcal{C} := \left\{ \hat{\mathbf{x}} \in \mathbb{R}^{\hat{n}} \mid \hat{\mathbf{x}}^\top \mathbf{Q} \hat{\mathbf{x}} < \bar{e}^2 \right\} \quad (11a)$$

$$\mathcal{D} := \left\{ \hat{\mathbf{x}} \in \mathbb{R}^{\hat{n}} \mid \hat{\mathbf{x}}^\top \mathbf{Q} \hat{\mathbf{x}} = \bar{e}^2 \right\}. \quad (11b)$$

In summary, Eqs. (7)–(11) are an impulsive system formulation of the investigated event-based state-feedback loop (3)–(6). The structure of this impulsive system is depicted in Fig. 2 which illustrates that the system  $\Sigma_c$  and  $\Sigma_d$  are interconnected in a cascade.

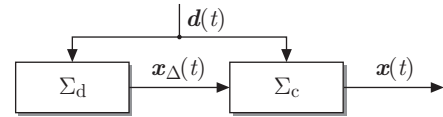


Fig. 2. Structure of the impulsive system (7)–(11)

## III. LYAPUNOV-BASED STABILITY ANALYSIS OF THE EVENT-BASED STATE-FEEDBACK LOOP

The cascade structure of the transformed event-based state-feedback loop (7)–(11) is exploited in the stability analysis method developed in this section, where the solutions  $\mathbf{x}(t)$  and  $\mathbf{x}_\Delta(t)$  of (7) or (8), respectively, are investigated separately. The boundedness of both  $\mathbf{x}(t)$  and  $\mathbf{x}_\Delta(t)$  implies the boundedness of the overall system state  $\hat{\mathbf{x}}(t)$ .

### A. Boundedness of the difference state

The initial reset of the difference state  $\mathbf{x}_\Delta$  at time  $t_0 = 0$  yields  $\mathbf{x}_\Delta(0) = \mathbf{0}$ . Hence, the relation

$$\|\mathbf{x}_\Delta(t)\|^2 \leq \bar{e}^2, \quad \forall t \geq 0 \quad (12)$$

directly follows from the event condition (10) and the state reset defined in (8).

## B. Boundedness of the plant state

Consider the system (7). The state  $\mathbf{x}(t)$  is bounded if (7) admits an ISS-Lyapunov function. A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is an ISS-Lyapunov function candidate for (7) with rate coefficient  $a_v > 0$  and scalars  $b_v, c_v > 0$  if  $V$  is positive definite, radially unbounded, and satisfies the inequality

$$\dot{V}(\mathbf{x}(t)) \leq -a_v V(\mathbf{x}(t)) + b_v \|\mathbf{d}(t)\|^2 + c_v \|\mathbf{x}_\Delta(t)\|^2 \quad (13)$$

for all  $\mathbf{x}, \mathbf{x}_\Delta \in \mathbb{R}^n$ ,  $\mathbf{d} \in \mathbb{R}^p$  and  $t \geq 0$ , [12].

The following investigation shows that

$$V(\mathbf{x}(t)) = \mathbf{x}^\top(t) \mathbf{P} \mathbf{x}(t), \quad \mathbf{P} = \mathbf{P}^\top \succ 0 \quad (14)$$

qualifies as an ISS-Lyapunov function for (7) and that parameters  $a_v, b_v, c_v > 0$  can always be found such that (13) is satisfied. Note that (13) with (7) and (14) has an equivalent representation by the following linear matrix inequality (LMI):

$$\begin{pmatrix} \bar{\mathbf{A}}^\top \mathbf{P} + \mathbf{P} \bar{\mathbf{A}} + a_v \mathbf{P} & \mathbf{P} \mathbf{E} & \mathbf{P} \mathbf{B} \mathbf{K} \\ \star & -b_v \mathbf{I} & \mathbf{0} \\ \star & \star & -c_v \mathbf{I} \end{pmatrix} \leq 0. \quad (15)$$

In order to investigate the feasibility of (15), first consider the LMI

$$\mathbf{M} := \begin{pmatrix} \bar{\mathbf{A}}^\top \mathbf{P} + \mathbf{P} \bar{\mathbf{A}} + a_v \mathbf{P} & \mathbf{P} \mathbf{E} \\ \star & -b_v \mathbf{I} \end{pmatrix} \prec 0. \quad (16)$$

The application of the Schur complement (see [4]) yields the following implication:

$$\mathbf{M} \prec 0 \Leftrightarrow \begin{cases} \bar{\mathbf{A}}^\top \mathbf{P} + \mathbf{P} \bar{\mathbf{A}} + a_v \mathbf{P} + b_v^{-1} \mathbf{P} \mathbf{E} \mathbf{E}^\top \mathbf{P} \prec 0 \\ -b_v \mathbf{I} \prec 0. \end{cases}$$

Note that there always exists a symmetric matrix  $\mathbf{P} \succ 0$  and a constant  $a_v > 0$  such that

$$\bar{\mathbf{A}}^\top \mathbf{P} + \mathbf{P} \bar{\mathbf{A}} + a_v \mathbf{P} \prec 0$$

holds, since  $\bar{\mathbf{A}}$  is Hurwitz. Even if  $\mathbf{P} \mathbf{E} \mathbf{E}^\top \mathbf{P}$  is generally indefinite, choosing  $b_v > 0$  large enough satisfies both LMIs on the right-hand side of the implication. Now, the feasibility of (15) follows by the same arguments. Using the Schur complement, the implication

$$(15) \Leftrightarrow \begin{cases} \mathbf{M} + c_v^{-1} \bar{\mathbf{P}} \bar{\mathbf{P}}^\top \leq 0, \\ -c_v \mathbf{I} \leq 0 \end{cases} \quad (17)$$

is obtained with  $\mathbf{M}$  defined in (16) and

$$\bar{\mathbf{P}} := \begin{pmatrix} \mathbf{P} \mathbf{B} \mathbf{K} \\ \mathbf{0} \end{pmatrix}.$$

Given  $\mathbf{M} \prec 0$ , there always exists some constant  $c_v > 0$  such that the LMIs on the right-hand side of the implication (17) are fulfilled.

In summary, this analysis showed that there exist parameters  $a_v, b_v, c_v > 0$  and a matrix  $\mathbf{P} \succ 0$  such that the function  $V$  defined in (14) is an ISS-Lyapunov function, because the matrix  $\bar{\mathbf{A}}$  is Hurwitz. This result implies the boundedness of the state  $\mathbf{x}(t)$ . The remaining part of this section shows how the function  $V$  can be used to determine an explicit bound on  $\mathbf{x}(t)$ .

Consider Eq. (13). Using the comparison lemma (see [5]), from Eq. (13) the inequality

$$V(\mathbf{x}(t)) \leq e^{-a_v t} V(\mathbf{x}_0) + \int_0^t e^{-a_v(t-\tau)} b_v \|\mathbf{d}(\tau)\|^2 d\tau + \int_0^t e^{-a_v(t-\tau)} c_v \|\mathbf{x}_\Delta(\tau)\|^2 d\tau, \quad \forall t \geq 0$$

follows. With the bounds (4), (12) the last inequality yields

$$V(\mathbf{x}(t)) \leq e^{-a_v t} V(\mathbf{x}_0) + \frac{b_v}{a_v} \bar{d}^2 + \frac{c_v}{a_v} \bar{e}^2, \quad \forall t \geq 0. \quad (18)$$

A bound on  $\mathbf{x}(t)$  can be derived from Eq. (18) as follows: Since  $V(\mathbf{x}) = \mathbf{x}^\top \mathbf{P} \mathbf{x}$  is positive definite and radially unbounded, it satisfies the relation

$$\underline{\alpha}(\|\mathbf{x}\|) \leq V(\mathbf{x}) \leq \bar{\alpha}(\|\mathbf{x}\|), \quad \forall \mathbf{x} \in \mathbb{R}^n \quad (19)$$

with

$$\underline{\alpha}(r) := \lambda_{\min}(\mathbf{P}) \cdot r^2, \quad \bar{\alpha}(r) := \lambda_{\max}(\mathbf{P}) \cdot r^2. \quad (20)$$

Hence, Eqs. (18)–(20) imply

$$\|\mathbf{x}(t)\| \leq \underline{\alpha}^{-1} \left( e^{-a_v t} \bar{\alpha}(\|\mathbf{x}_0\|) \right) + \underline{\alpha}^{-1} \left( \frac{b_v}{a_v} \bar{d}^2 \right) + \underline{\alpha}^{-1} \left( \frac{c_v}{a_v} \bar{e}^2 \right), \quad \forall t \geq 0. \quad (21)$$

The results obtained in this section are summarized in the following lemma.

*Lemma 1:* Given that the matrix  $\bar{\mathbf{A}}$  is Hurwitz, there always exists an ISS-Lyapunov function (14) for the system (7) and parameters  $a_v, b_v, c_v > 0$  satisfying (13) which can be determined using the LMI (15). The state  $\mathbf{x}(t)$  of (7) is bounded according to (21).

## C. Stability of the event-based state-feedback loop

The results of the previous two sections are now combined in order to arrive at a bound on the state  $\hat{\mathbf{x}}(t)$  for the event-based state-feedback loop (7)–(11). Note that the relation

$$\|\hat{\mathbf{x}}(t)\| \leq \|\mathbf{x}(t)\| + \|\mathbf{x}_\Delta(t)\|, \quad \forall t \geq 0 \quad (22)$$

holds. Hence, Eq. (22) together with the bounds (12), (21) implies the estimate (2) with

$$\beta(r, t) := \underline{\alpha}^{-1} \left( e^{-a_v t} \bar{\alpha}(r) \right) \quad (23a)$$

$$\gamma(r) := \underline{\alpha}^{-1} \left( \frac{b_v}{a_v} r^2 \right) \quad (23b)$$

$$\sigma := \underline{\alpha}^{-1} \left( \frac{c_v}{a_v} \bar{e}^2 \right) + \bar{e}. \quad (23c)$$

The presented Lyapunov-based stability analysis is summarized in the following theorem.

*Theorem 1:* The event-based state-feedback loop (7)–(11) is ISpS and the state  $\hat{\mathbf{x}}(t)$  is bounded according to Eqs. (2), (23).

#### D. Discussion of the analysis result

The result of the stability analysis, summarized in Theorem 1 can be rephrased as follows: Consider the continuous state-feedback loop

$$\dot{\mathbf{x}}_{\text{SF}}(t) = \bar{\mathbf{A}}\mathbf{x}_{\text{SF}}(t) + \mathbf{E}\mathbf{d}(t), \quad \mathbf{x}_{\text{SF}}(0) = \mathbf{x}_0, \quad (24)$$

with state  $\mathbf{x}_{\text{SF}} \in \mathbb{R}^n$  and  $\bar{\mathbf{A}} = (\mathbf{A} - \mathbf{B}\mathbf{K})$ . Since  $\bar{\mathbf{A}}$  is Hurwitz, (24) is ISS. Hence, ISS of the continuous state-feedback loop (24) implies ISpS of the event-based state-feedback loop (7)–(11).

This implication basically corresponds to analysis results known from literature on event-based control. Reference [9] has proven that the difference between the behavior of the continuous state-feedback loop (24) and the event-based state-feedback loop (3)–(6) is bounded and this bound is linearly dependent upon the event threshold  $\bar{\epsilon}$ . A different Lyapunov-based approach to the stability analysis has been proposed in [6] where the state  $\mathbf{x}(t)$  of the system (7) was shown to be ultimately bounded, i. e. there exists some time  $\bar{t}$  such that

$$\mathbf{x}(t) \in \mathcal{B}_c := \{\mathbf{x} \mid \|\mathbf{x}\| \leq \varphi_c\}, \quad \forall t \geq \bar{t}$$

where  $\varphi_c$  is referred to as the *ultimate bound* of the system (7). Observe that in (2) the term

$$\varphi(r) = \gamma(r) + \sigma \quad (25)$$

can be interpreted as an ultimate bound of the event-based state-feedback loop (7)–(11) because  $\|\hat{\mathbf{x}}(t)\| \rightarrow \varphi$  as  $t \rightarrow \infty$  for arbitrary initial conditions  $\hat{\mathbf{x}}_0$ , [11].

The mentioned references and the new stability analysis method summarized in Theorem 1 have in common that the event-based state-feedback loop is proven to be ISpS. That is, all these analysis methods result in the fact that there is a term  $\sigma > 0$  contributing to the ultimate bound  $\varphi$  which is constant and does not depend upon the disturbance bound  $\bar{d}$ . The stability analysis presented in Sec. IV will show that this result is conservative and not true for event-based state-feedback loops (7)–(11) with stable plant dynamics.

#### E. Comparison to an existing analysis method

The impulsive system formulation of an event-based control loop with state feedback and an analysis based on this representation has been introduced in [2]. However, the structure of the event-based control loop investigated in this reference differs from the one that is studied here. In place of the control input generator (5) a ZOH has been applied which results in the model (1) with

$$\hat{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{A}} & \mathbf{B}\mathbf{K} \\ \bar{\mathbf{A}} & \mathbf{B}\mathbf{K} \end{pmatrix}, \quad \hat{\mathbf{E}} = \begin{pmatrix} \mathbf{E} \\ \mathbf{E} \end{pmatrix}, \quad \hat{\mathbf{G}} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

instead of (9). The structure of the system investigated in [2] is illustrated in Fig. 3 where  $\tilde{\Sigma}_d$  represents the difference system with changed dynamics compared to the system (8). The transformed system has a feedback that does not allow for a sequential analysis of the systems  $\tilde{\Sigma}_d$  and  $\Sigma_c$  as done in this paper, but requires a different analysis approach that incorporates the entire system.

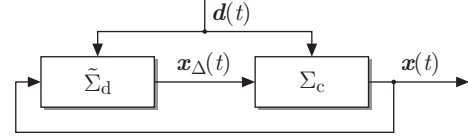


Fig. 3. Structure of the transformed event-based control loop with ZOH

#### IV. ANALYSIS OF THE EVENT-BASED CONTROL LOOP WITH STABLE PLANT DYNAMICS

This section develops a novel analysis method for event-based state-feedback loops (7)–(11) with stable plants.

*Assumption 1:* The plant (3) is ISS, i. e. the matrix  $\mathbf{A}$  is Hurwitz.

The aim of this analysis is to show that the event-based state-feedback loop (7)–(11) with stable plant dynamics is ISS rather than ISpS, which is a new result concerning event-based control schemes with constant event threshold.

##### A. Boundedness of the difference state

This section refines the result on the boundedness of the difference state  $\mathbf{x}_\Delta(t)$  presented in Sec. III-A. The bound to be developed is stated in terms of the positive definite and radially unbounded function  $W : \mathbb{R}^n \rightarrow \mathbb{R}$  with

$$W(\mathbf{x}_\Delta(t)) := \|\mathbf{x}_\Delta(t)\|^2 = \mathbf{x}_\Delta^\top(t)\mathbf{I}\mathbf{x}_\Delta(t),$$

which is assumed to satisfy the relation

$$\dot{W}(\mathbf{x}_\Delta(t)) \leq -a_w W(\mathbf{x}_\Delta(t)) + b_w \|\mathbf{d}(t)\|^2 \quad (26)$$

for some  $a_w, b_w > 0$  and for all  $t \in [t_k, t_{k+1})$ .

As explained hereafter, appropriate parameters  $a_w, b_w > 0$  always exist such that Eq. (26) holds, given that Assumption 1 is fulfilled. Note that (26) with (8) yields

$$\frac{dW(\mathbf{x}_\Delta)}{d\mathbf{x}_\Delta} (\mathbf{A}\mathbf{x}_\Delta(t) + \mathbf{E}\mathbf{d}(t)) \leq -a_w W(\mathbf{x}_\Delta(t)) + b_w \|\mathbf{d}(t)\|^2,$$

which can be restated as the following LMI:

$$\begin{pmatrix} \mathbf{A}^\top + \mathbf{A} + a_w \mathbf{I} & \mathbf{E} \\ \star & -b_w \mathbf{I} \end{pmatrix} \preceq \mathbf{0}. \quad (27)$$

Using the Schur complement the implication

$$(27) \Leftrightarrow \begin{cases} \mathbf{A}^\top + \mathbf{A} + a_w \mathbf{I} + b_w^{-1} \mathbf{E}\mathbf{E}^\top \preceq \mathbf{0} \\ -b_w \mathbf{I} \preceq \mathbf{0} \end{cases} \quad (28)$$

is obtained. Given that Assumption 1 holds there always exists some  $a_w > 0$  such that

$$\mathbf{A}^\top + \mathbf{A} + a_w \mathbf{I} \prec \mathbf{0}.$$

Hence, choosing  $b_w > 0$  large enough satisfies both LMIs on the right-hand side of the implication (28).

Now, Eq. (26) is used to determine a bound on the difference state  $\mathbf{x}_\Delta(t)$  in terms of the function  $W$ . Recall that according to (8) a reset at the event times  $t_k$  results in the fact that  $\mathbf{x}_\Delta(t_k^+) = \mathbf{0}$ , which implies  $W(\mathbf{x}_\Delta(t_k^+)) = \mathbf{0}$ . Hence, the application of the comparison lemma to (26) yields

$$W(\mathbf{x}_\Delta(t)) \leq \int_{t_k}^t e^{-a_w(t-\tau)} b_w \|\mathbf{d}(\tau)\|^2 d\tau$$

for  $t \in [t_k, t_{k+1})$ . Observe that the right-hand side of the last inequality is bounded from above by

$$W(\mathbf{x}_\Delta(t)) \leq \int_0^\infty e^{-a_w \tau} d\tau b_w \bar{d}^2 = \frac{b_w}{a_w} \bar{d}^2 \quad (29)$$

for all  $t \geq 0$ . Equation (29) represents a bound on the difference state  $\mathbf{x}_\Delta(t)$  in dependence upon the disturbance magnitude  $\bar{d}$  which, however, may exceed the value  $\bar{e}^2$  for large disturbances. In virtue of the event condition (10) and the state reset defined in (8) the estimate (29) can be made more precise:

$$W(\mathbf{x}_\Delta(t)) \leq \min\left(\frac{b_w}{a_w} \bar{d}^2, \bar{e}^2\right) =: \hat{W}(\bar{d}), \quad \forall t \geq 0. \quad (30)$$

This result on the boundedness of the difference state  $\mathbf{x}_\Delta(t)$  is summarized in the following lemma.

*Lemma 2:* Consider the difference system (8) and let Assumption 1 hold. Then the state  $\mathbf{x}_\Delta(t)$  of (8) is bounded by

$$\|\mathbf{x}_\Delta(t)\|^2 \leq \hat{W}(\bar{d}), \quad \forall t \geq 0, \quad (31)$$

where  $\hat{W}(\bar{d})$  is defined in Eq. (30).

In contrast to Eq. (12) the bound (31) takes account of the fact that a deviation of the difference state  $\mathbf{x}_\Delta(t)$  from the origin is only caused by a disturbance  $\mathbf{d}(t)$ . Hence, Eq. (30) also reflects that  $\|\mathbf{x}_\Delta(t)\|$  never reaches the event threshold  $\bar{e}$ , i. e. no events are triggered, if the disturbance magnitude  $\bar{d}$  is small, which means  $\bar{d}$  satisfies the relation

$$\bar{d} < \bar{e} \cdot \left(\frac{b_w}{a_w}\right)^{-1/2} =: \delta. \quad (32)$$

### B. Boundedness of the plant state

Following the analysis presented in Sec. III-B, this section determines a bound on the state  $\mathbf{x}(t)$  of the system (7) by means of the ISS-Lyapunov function  $V(\mathbf{x}(t))$  defined in (14) which has been shown to satisfy the inequality (13) for appropriate  $\mathbf{P} \succ 0$  and parameters  $a_v, b_v, c_v > 0$ .

Applying the comparison lemma to Eq. (13) yields

$$V(\mathbf{x}(t)) \leq e^{-a_v t} V(\mathbf{x}_0) + \int_0^t e^{-a_v(t-\tau)} b_v \|\mathbf{d}(\tau)\|^2 d\tau + \int_0^t e^{-a_v(t-\tau)} c_v \|\mathbf{x}_\Delta(\tau)\|^2 d\tau, \quad \forall t \geq 0.$$

By substituting (4) and (31) into the last inequality, a bound on  $V(\mathbf{x}(t))$  is given by

$$V(\mathbf{x}(t)) \leq e^{-a_v t} V(\mathbf{x}_0) + g(\bar{d}), \quad \forall t \geq 0 \quad (33)$$

with the function  $g: \mathbb{R} \rightarrow \mathbb{R}$ ,  $g \in \mathcal{K}_\infty$

$$g(\bar{d}) := \frac{1}{a_v} (b_v \bar{d}^2 + c_v \hat{W}(\bar{d})) \quad (34)$$

An estimate on the state  $\mathbf{x}(t)$  can be obtained from Eqs. (33), (34) by proceeding according to Sec. III-B:  $V(\mathbf{x}(t))$  satisfies the relation (19) for some functions  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$  defined in (20). Thus, Eq. (33) implies

$$\|\mathbf{x}(t)\| \leq \underline{\alpha}^{-1}\left(e^{-a_v t} \bar{\alpha}(\|\mathbf{x}_0\|)\right) + \underline{\alpha}^{-1}(g(\bar{d})) \quad (35)$$

for all  $t \geq 0$ .

The following lemma summarizes the result on the boundedness of the state  $\mathbf{x}(t)$ .

*Lemma 3:* Consider the system (7) and let Assumption 1 hold. Then the state  $\mathbf{x}(t)$  is bounded according to (35), where  $\mathbf{P} \succ 0$  of (14) and the parameters  $a_v, b_v, c_v > 0$  can be determined using the LMI (15) and the functions  $\underline{\alpha}, \bar{\alpha}$  and  $g(\bar{d})$  defined in (20) or (34), respectively.

### C. Stability of the event-based state-feedback loop

A bound on the state  $\hat{\mathbf{x}}(t)$  of the event-based state-feedback loop (7)–(11) with stable plant dynamics is now obtained using the relation (22). Equation (22) and the bounds (31), (35) yield the estimate (2) with

$$\beta(r, t) := \underline{\alpha}^{-1}\left(e^{-a_v t} \bar{\alpha}(r)\right) \quad (36a)$$

$$\gamma(r) := \underline{\alpha}^{-1}(g(r)) + (\hat{W}(r))^{1/2} \quad (36b)$$

$$\sigma = 0. \quad (36c)$$

Note that the function  $\beta(r, t)$  in (36a) is the same as in Eq. (23a). Since  $\sigma = 0$  holds, the event-based state-feedback loop (7)–(11) with stable plant dynamics is ISS.

*Theorem 2:* If the plant is stable, i. e. the matrix  $\mathbf{A}$  is Hurwitz, the event-based state-feedback loop (7)–(11) is ISS and the state  $\hat{\mathbf{x}}(t)$  is bounded according to Eqs. (2), (36).

Theorem 2 implies asymptotic convergence of the state  $\hat{\mathbf{x}}(t)$  to the origin if the disturbance  $\mathbf{d}(t)$  vanishes. This result is known from literature only for event-based control schemes that use a decreasing event threshold, e. g. in [3], [10], [14]–[16], but not for event-based controllers with constant event threshold as investigated in this paper.

## V. EXAMPLE

### A. Thermofluid process

The presented analysis methods are now illustrated using the example of a thermofluid process [7]. The plant is described by the linearized model (3) with

$$\mathbf{A} = 10^{-3} \begin{pmatrix} -0.8 & 0 \\ -1 \cdot 10^{-7} & -1.7 \end{pmatrix} \quad (37)$$

$$\mathbf{B} = \begin{pmatrix} 0.21 & 0 \\ -0.11 & 0.02 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 0.15 \\ -0.08 \end{pmatrix}.$$

The event-based controller consists of the control input generator (5) with the state-feedback gain

$$\mathbf{K} = \begin{pmatrix} 0.08 & -0.02 \\ 0.17 & 0.72 \end{pmatrix}$$

and the event generator which uses the condition (6) with event threshold  $\bar{e} = 2$  to determine the event times. A comparison of the analysis methods is accomplished by means of the ultimate bound  $\varphi$  of the event-based state-feedback loop (7)–(11) defined in Eq. (25).

### B. Analysis method according to Sec. III

For the considered example, the function (14) with

$$\mathbf{P} = 10^{-3} \begin{pmatrix} 45.1 & 3.1 \\ 3.1 & 48.2 \end{pmatrix} \quad (38)$$

satisfies the relation (13) for the parameters

$$a_v = 0.015, \quad b_v = 0.4, \quad c_v = 1 \cdot 10^{-3}. \quad (39)$$

From Eq. (20) with (38) the comparison functions

$$\underline{\alpha}(r) = r^2 \cdot 43.2 \cdot 10^{-3} \Leftrightarrow \underline{\alpha}^{-1}(r) = 4.81 \cdot \sqrt{r}$$

$$\bar{\alpha}(r) = r^2 \cdot 50.2 \cdot 10^{-3}$$

follow. According to Eq. (23) these results yield

$$\beta(r, t) = 1.08 \cdot r \cdot e^{-\frac{0.015}{2}t}, \quad \gamma(r) = 24.8 \cdot r, \quad \sigma = 4.48.$$

Hence, the ultimate bound  $\varphi$  that is obtained by means of this analysis method adds up to

$$\varphi(r) = \gamma(r) + \sigma = 24.8 \cdot r + 4.48.$$

### C. Analysis method according to Sec. IV

The second proposed analysis method is applicable to the benchmark, since the matrix  $\mathbf{A}$  stated in (37) is Hurwitz. First the function  $\hat{W}$  is determined as specified in (30). Therefore, observe that for this example the inequality (26) is fulfilled for the parameters

$$a_w = 1 \cdot 10^{-3}, \quad b_w = 41,$$

which yields

$$\hat{W}(r) = \min(4.1 \cdot 10^4 \cdot r^2, 4).$$

In order to determine the functions  $\beta$  and  $\gamma$  as specified in (36), the ISS-Lyapunov function (14), (38) and the parameters given in (39) yield

$$\begin{aligned} \beta(r, t) &= 1.08 \cdot r \cdot e^{-\frac{0.015}{2}t} \\ \gamma(r) &= 4.81 \cdot (26.7 \cdot r^2 + 0.07 \cdot \min(4.1 \cdot 10^4 \cdot r^2, 4))^{1/2} \\ &\quad + \min(202 \cdot r, 2). \end{aligned} \quad (40)$$

Since  $\sigma = 0$  holds, these results amount to the ultimate bound  $\varphi(r) = \gamma(r)$  as given in (40). Finally, from Eq. (32)

$$\delta = 0.01 \quad (41)$$

follows, where  $\delta$  is the largest disturbance magnitude for which no events are generated.

### D. Comparison of the analysis results

Figure 4 illustrates the analysis results obtained with the methods proposed in Sec. III and Sec. IV and compares them with the result of [6] as well as the actual ultimate bound of the event-based state-feedback loop (7)–(11). The figure shows the ultimate bound  $\varphi$  plotted against the disturbance magnitude  $\bar{d}$ .

The analysis method proposed in [6] and the novel method presented in Sec. III yield similar results. Both methods prove the event-based state-feedback loop (7)–(11) to be ISpS and, thus, obtain an ultimate bound  $\varphi$  that has an offset at  $\bar{d} = 0$ . In contrast to this, the analysis method presented in Sec. IV yields an ultimate bound  $\varphi$  that reflects the characteristics of the actual ultimate bound. The bound grows linearly with  $\bar{d}$  in the interval  $\bar{d} \in [0, \delta]$  with  $\delta$  as is (41). The effect of state reinitialization is taken into account for all  $\bar{d} > \delta$  which reduces the slope of the curve. The estimate is considerably less conservative compared to the results of the previously discussed analysis methods. However, the derived disturbance magnitude  $\delta$  up to which no events are triggered is by one third smaller than the actual value  $\delta_{\text{act}} = 0.03$  which has been determined by means of simulations.

In summary, this comparison shows that a more detailed analysis of the difference state  $x_{\Delta}(t)$  as done in Sec. IV results in a more accurate estimate on the actual ultimate bound of the event-based state-feedback loop (7)–(11) with stable plant dynamics, than the constant bound (12) that has been used in the analysis presented in Sec. III and methods known from literature.

## VI. CONCLUSION

The paper proposed two new methods for the stability analysis of event-based state-feedback loops based on the representation of the control loop as an impulsive system. The investigations have shown that the event-based state-feedback loop is generally ISpS which is in line with the analysis results known from literature. However, if the plant

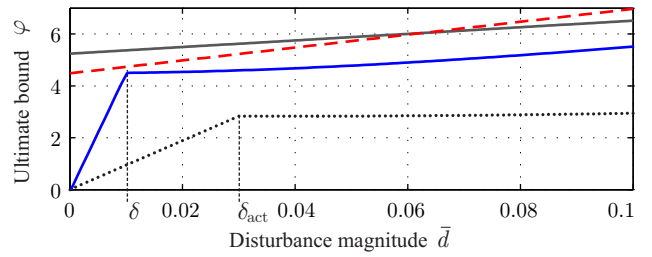


Fig. 4. Comparison of the analysis results

dynamics are stable, the second analysis method proved the event-based control loop to be ISS. This result implies asymptotic stability if the disturbance vanishes. The analysis methods have been compared with respect to the ultimate bound, using the example of a thermofluid benchmark process. This comparison showed that the method which is tailored for event-based state-feedback loops with stable plant dynamics considerably diminishes the conservatism of existing methods known from literature.

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